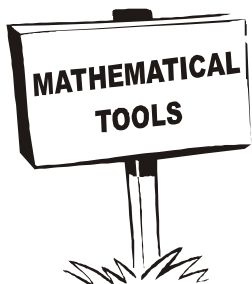# MATHEMATICAL TOOLS

Mathematics is the language of physics. It becomes easier to describe, understand and apply the physical principles, if one has a good knowledge of mathematics.

MATHEMATICAL TOOLS

⑩ **Tools are required to do physical work easily and mathematical tools are required to solve numerical problems easily.**

MATHEMATICAL TOOLS
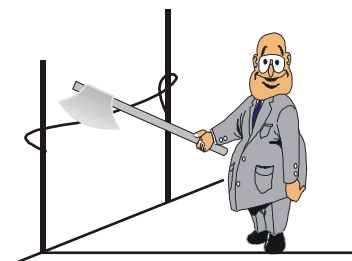
Differentiation	Integration	Vectors

To solve the problems of physics Newton made significant contributions to Mathematics by inventing differentiation and integration.

Cutting a tree with a blade

Cutting a string with an axe

APPROPRIATE CHOICE OF TOOL IS VERY IMPORTANT

1. FUNCTION

Function is a rule of relationship between two variables in which one is assumed to be dependent and the other independent variable, for example :

e.g. The temperatures at which water boils depends on the elevation above sea level (the boiling point drops as you ascend). Here elevation above sea level is the independent & temperature is the dependent variable

e.g. The interest paid on a cash investment depends on the length of time the investment is held. Here time is the independent and interest is the dependent variable.

In each of the above example, value of one variable quantity (dependent variable) , which we might call y, depends on the value of another variable quantity (independent variable), which we might call x. Since the value of y is completely determined by the value of x, we say that y is a function of x and represent it mathematically as y = f(x).

Here f represents the function, x the independent variable & y is the dependent variable.

All possible values of independent variables (x) are called **domain** of function.

All possible values of dependent variable (y) are called **range** of function.

Think of a function f as a kind of machine that produces an output value f(x) in its range whenever we feed it an input value x from its domain (figure).

When we study circles, we usually call the area A and the radius r. Since area depends on radius, we say that A is a function of r, A = f(r). The equation $A = \pi r^2$ is a rule that tells how to calculate a unique (single) output value of A for each possible input value of the radius r.

$A = f(r) = \pi r^2$. (Here the rule of relationship which describes the function may be described as square & multiply by π).

If r = 1   A = π ;   if r = 2 A = 4π ;   if r = 3   A = 9π

The set of all possible input values for the radius is called the domain of the function. The set of all output   values of the area is the range of the function.

We usually denote functions in one of the two ways :

1. By giving a formula such as y = x^2 that uses a dependent variable y to denote the value of the function.

2. By giving a formula such as f(x) = x^2 that defines a function symbol f to name the function.

   Strictly speaking, we should call the function f and not f(x),

   y = sin x. Here the function is sine, x is the independent variable.

Solved Example

Example 1. The volume V of a ball (solid sphere) of radius r is given by the function $V(r) = (4/3)\pi(r)^3$. The volume of a ball of radius 3m is ?

Solution : $V(3) = 4/3\pi(3)^3 = 36\pi$ m^3.

Example 2. Suppose that the function F is defined for all real numbers r by the formula $F(r) = 2(r − 1) + 3$. Evaluate F at the input values 0, 2, x + 2, and F(2).

Solution : In each case we substitute the given input value for r into the formula for F :

$F(0) = 2(0 − 1) + 3 = − 2 + 3 = 1$;

$F(2) = 2(2 − 1) + 3 = 2 + 3 = 5$

$F(x + 2) = 2(x +2 − 1) + 3 = 2x + 5$;

$F(F(2)) = F(5) = 2(5 − 1) + 3 = 11$.

Example 3. A function $f(x)$ is defined as $f(x) = x^2 + 3$, Find $f(0)$, $f(1)$, $f(x^2)$, $f(x+1)$ and $f(f(1))$.

Solution : $f(0) = 0^2 + 3 = 3$; $f(1) = 1^2 + 3 = 4$; $f(x^2) = (x^2)^2+3 = x^4+3$

$f(x+1) = (x + 1)^2 + 3 = x^2 + 2x + 4$; $f(f(1)) = f(4) = 4^2+3 = 19$

Example 4. If function F is defined for all real numbers x by the formula $F(x) = x^2$. Evaluate F at the input values 0, 2, x + 2 and F(2)

Solution : $F(0) = 0$; $F(2) = 2^2 = 4$; $F(x+2) = (x+2)^2$
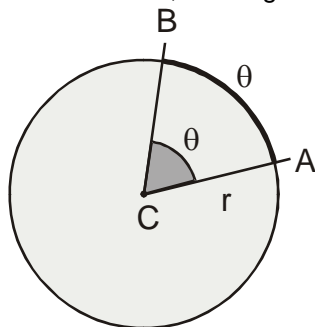
$F(F(2)) = F(4) = 4^2 = 16$

2. TRIGONOMETRY

2.1. Measurement of angle and relationship between degrees and radian

In navigation and astronomy, angles are measured in degrees, but in calculus it is best to use units called radians because of the way they simplify later calculations.

Let ACB be a central angle in a **circle** of radius r, as in figure.

Then the angle ACB or θ is defined in radius as θ = $\dfrac{\text{Arc length}}{\text{Radius}}$ $\Rightarrow$ θ = $\dfrac{\overset{\frown}{AB}}{r}$

If r = 1 then θ = AB

The **radian measure** for a circle of unit radius of angle ACB is defined to be the length of the circular arc AB. Since the circumference of the circle is 2π and one complete revolution of a circle is 360º, the relation between radians and degrees is given by : π radians = 180º

Angle Conversion formulas

1 degree = $\dfrac{\pi}{180}$ ($\approx$ 0.02) radian

Degrees to radians : multiply by $\dfrac{\pi}{180}$

1 radian $\dfrac{\pi}{180}$ $\approx$ 57 degrees

Radians to degrees : multiply by $\dfrac{180}{\pi}$

Solved Example

Example 1.   (i) Convert 45º to radians.   (ii) Convert $\dfrac{\pi}{6}$ rad to degrees.

Solution :   (i) $45 \cdot \dfrac{\pi}{180} = \dfrac{\pi}{4}$ rad   (ii) $\dfrac{\pi}{6} \cdot \dfrac{180}{\pi} = 30º$

Example 2.   Convert 30º to radians.

Solution : $30º \times \dfrac{\pi}{180} = \dfrac{\pi}{6}$ rad

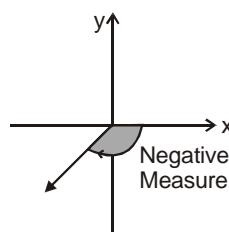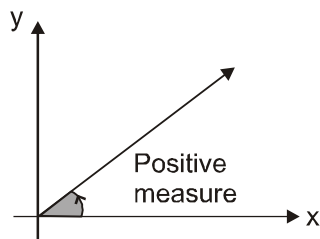**Example 3.**   Convert $\dfrac{\pi}{3}$ rad to degrees.

**Solution :** $\dfrac{\pi}{3} \times \dfrac{180}{\pi} = 60º$

Standard values

(1)	$30° = \dfrac{\pi}{6}$ rad	(2)	$45º = \dfrac{\pi}{4}$ rad	(3)	$60º =$ rad
(4)	$90º = \dfrac{\pi}{2}$ rad	(5)	$120º = \dfrac{2\pi}{3}$ rad	(6)	$135º =$ rad
(7)	$150º = \dfrac{5\pi}{6}$ rad	(8)	$180º = \pi$ rad	(9)	$360º = 2\pi$ rad

## 2.2. Measurement of positive and Negative Angles

An angle in the xy-plane is said to be in standard position if its vertex lies at the origin and its initial ray lies along the positive x-axis (Fig.). Angles measured counterclockwise from the positive x-axis are assigned positive measures; angles measured clockwise are assigned negative measures.
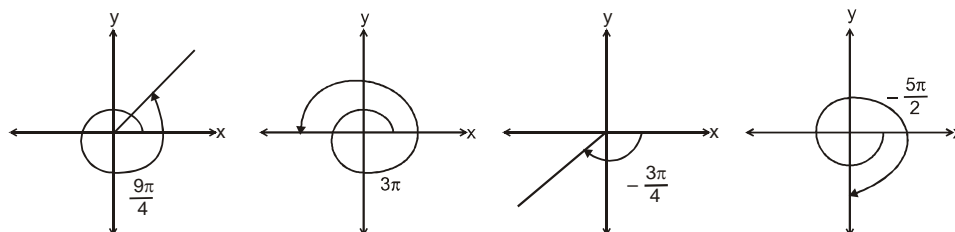
2.3. Six Basic Trigonometric Functions

The trigonometric function of a general angle θ are defined in terms of x, y, and r.
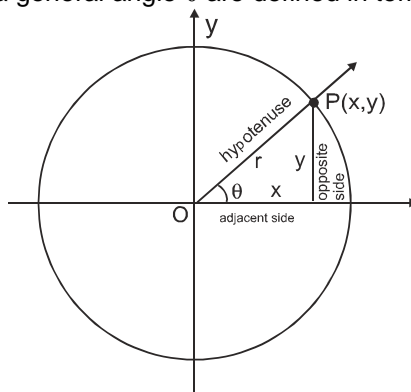
Sine : $\sin\theta = \dfrac{opp}{hyp} = \dfrac{y}{r}$ Cosecant : $\csc\theta = \dfrac{hyp}{opp} = \dfrac{r}{y}$

Cosine : $\cos\theta = \dfrac{adj}{hyp} = \dfrac{x}{r}$ Secant : $\sec\theta = \dfrac{hyp}{adj} = \dfrac{r}{x}$

Tangent : $\tan\theta = \dfrac{opp}{adj} = \dfrac{y}{x}$ Cotangent : $\cot\theta = \dfrac{adj}{opp} = \dfrac{x}{y}$

VALUES OF TRIGONOMETRIC FUNCTIONS

If the circle in (Fig. above) has radius r = 1, the equations defining $\sin\theta$ and $\cos\theta$ become

$\cos\theta = x$, $\sin\theta = y$

We can then calculate the values of the cosine and sine directly from the coordinates of P.

Solved Example

Example 1.   Find the six trigonometric ratios from given figure
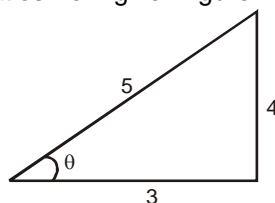
Solution : $\sin\theta = \dfrac{opp}{hyp} = \dfrac{4}{5}$; $\cos\theta = \dfrac{adj}{hyp} = \dfrac{3}{5}$; $\tan\theta = \dfrac{opp}{adj} = \dfrac{4}{3}$; $\csc\theta = \dfrac{hyp}{opp} = \dfrac{5}{4}$;

$\sec\theta = \dfrac{hyp}{adj} = \dfrac{5}{3}$; $\cot\theta = \dfrac{adj}{opp} = \dfrac{3}{4}$

Example 2.   Find the sine and cosine of angle θ shown in the unit circle if coordinate of point p are as shown.
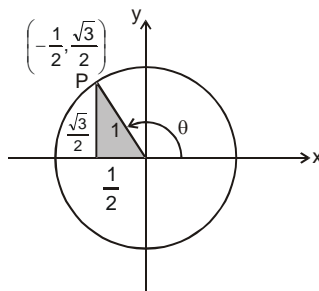
Solution : $\cos\theta$ = x-coordinate of P = $-\dfrac{1}{2}$; $\sin\theta$ = y-coordinate of P = $\dfrac{\sqrt{3}}{2}$.

2.4. Rules for finding Trigonometric ratio of angles greater than 90°

Step-1. Identify the quadrant in which angle lies.

Step-2. →

(a) If angle = (nπ ± θ) where n is an integer. Then trigonometric function of (nπ ± θ) = same trigonometric function of θ and sign will be decided by CAST Rule.

```
THE CAST RULE                    │ y
                                 │
                    II Quadrant  │  I Quadrant
A useful rule for remembering when the   S      │      A
basic trigonometric functions are positive   sin positive │ all positive
and negative is the CAST rule.           │
If you are not very enthusiastic about CAST. ────────────────┼──────────────→ x
You can remember it as ASTC          T      │      C
(After school to college)       tan positive │ cos positive
                                 │
                   III Quadrant  │  IV Quadrant
```

(b) If angle = $\left[(2n+1)\dfrac{\pi}{2} \pm \theta\right]$ where n is an integer. Then trigonometric function of $\left[(2n+1)\dfrac{\pi}{2} \pm \theta\right]$ = complimentary trignometric function of θ and sign will be decided by CAST Rule.

Degree	0	30	37	45	53	60	90	120	135	180
Radians	0	$\pi/6$	$37\pi/180$	$\pi/4$	$53\pi/180$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	π
$\sin\theta$	0	$1/2$	$3/5$	$1/\sqrt{2}$	$4/5$	$\sqrt{3}/2$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	0
$\cos\theta$	1	$\sqrt{3}/2$	$4/5$	$1/\sqrt{2}$	$3/5$	$1/2$	0	$-1/2$	$-1/\sqrt{2}$	-1
$\tan\theta$	0	$1/\sqrt{3}$	$3/4$	1	$4/3$	$\sqrt{3}$	∞	$-\sqrt{3}$	-1	0

Values of sin θ, cos θ and tan θ for some standard angles.

Solved Example

Example 1. Evaluate sin 120°

Solution : sin 120° = sin (90° + 30°) = cos 30° = $\dfrac{\sqrt{3}}{2}$

Aliter sin 120° = sin (180° − 60°) = sin 60° = $\dfrac{\sqrt{3}}{2}$

Example 2. Evaluate cos 135°

Solution : cos 135° = cos (90° + 45°) = − sin 45° = $-\dfrac{1}{\sqrt{2}}$

Example 3. Evaluate cos 210°

Solution : cos 210° = cos (180° + 30°) = − cos30° = $-\dfrac{\sqrt{3}}{2}$

Example 4. Evaluate tan 210°

Solution : tan 210° = tan (180° + 30°) = tan 30° = $\dfrac{1}{\sqrt{3}}$

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2.5. General Trigonometric Formulas :

1.
$$\cos^2 \theta + \sin^2 \theta = 1$$
$$1 + \tan^2 \theta = \sec^2 \theta.$$
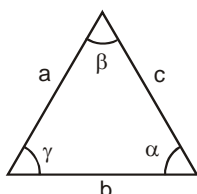$$1 + \cot^2 \theta = \operatorname{cosec}^2 \theta.$$

2.
$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$
$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$
$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

3. $\sin 2\theta = 2 \sin \theta \cos \theta$; $\cos 2\theta = \cos^2 \theta - \sin^2\theta = 2\cos^2 \theta - 1 = 1 - 2\sin^2 \theta$
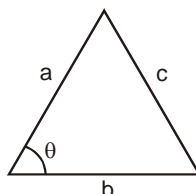
$\cos^2 \theta = \dfrac{1 + \cos 2\theta}{2}$; $\sin^2 \theta = \dfrac{1 - \cos 2\theta}{2}$

4. sine rule for triangles

5. cosine rule for triangles

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

$$c^2 = a^2 + b^2 - 2ab \cos\theta$$

3. DIFFERENTIATION

3.1. Finite difference

The finite difference between two values of a physical quantity is represented by Δ notation.

For example :

Difference in two values of y is written as Δy as given in the table below.

y_2	100	100	100
y_1	50	99	99.5
$\Delta y = y_2 - y_1$	50	1	0.5

INFINITELY SMALL DIFFERENCE :

The infinitely small difference means very-very small difference. And this difference is represented by 'd' notation instead of 'Δ'.

For example infinitely small difference in the values of y is written as 'dy'

if $y_2 = 100$ and $y_1 = 99.99999999........$

then dy = 0.000000...................00001

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ADVMT - 7

3.2. Definition of differentiation

Another name for differentiation is derivative. Suppose y is a function of x or y = f(x).

Differentiation of y with respect to x is denoted by symbol f′(x) where $f'(x) = \dfrac{dy}{dx}$

dx is very small change in x and dy is corresponding very small change in y.

NOTATION : There are many ways to denote the derivative of a function y = f(x). Besides f′(x), the most common notations are these :

y′	"y prime" or "y dash"	Nice and brief but does not name the independent variable.
$\dfrac{dy}{dx}$	"dy by dx"	Names the variables and uses d for derivative.
$\dfrac{df}{dx}$	"df by dx"	Emphasizes the function's name.
$\dfrac{d}{dx}f(x)$	"d by dx of f"	Emphasizes the idea that differentiation is an operation performed on f.
$D_x f$	"dx of f"	A common operator notation.
$\dot{y}$	"y dot"	One of Newton's notations, now common for time derivatives i.e. $\dfrac{dy}{dt}$.
f′(x)	f dash x	Most common notation, it names the independent variable and Emphasize the function's name.

3.3. Slope of a line

It is the tan of angle made by a line with the positive direction of x-axis, measured in anticlockwise direction.

Slope = tan θ   (In 1st quadrant tan θ is +ve & 2nd quadrant tan θ is –ve)
In Figure - 1 slope is positive   In Figure - 2 slope is negative
θ < 90° (1st quadrant)   θ > 90° (2nd quadrant)
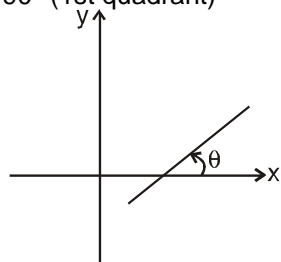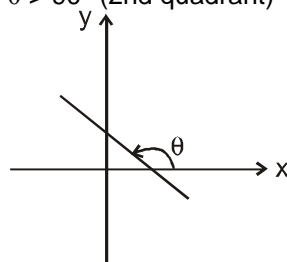
Figure - 1   Figure - 2

3.4. Average Rates of Change :

Given an arbitrary function y = f(x) we calculate the average rate of change of y with respect to x over the interval (x, x + Δx) by dividing the change in value of y, i.e. Δy = f(x + Δx) – f(x), by length of interval Δx over which the change occurred.

The average rate of change of y with respect to x over the interval [x, x + Δx] = $\dfrac{\Delta y}{\Delta x} = \dfrac{f(x+\Delta x)-f(x)}{\Delta x}$

In triangle QPR $\tan\theta = \dfrac{\Delta y}{\Delta x}$

Geometrically, $\dfrac{\Delta y}{\Delta x} = \dfrac{QR}{PR}$ = tan θ = Slope of the line PQ

therefore we can say that average rate of change of y with respect to x is equal to slope of the line joining P & Q.
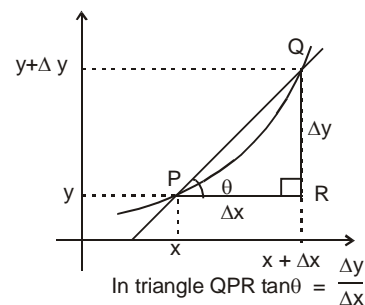
3.5. The derivative of a function

We know that, average rate of change of y w.r.t. x is $\dfrac{\Delta y}{\Delta x} = \dfrac{f(x+\Delta x)-f(x)}{\Delta x}$. If the limit of this ratio exists as $\Delta x \rightarrow 0$, then it is called the derivative of given function f(x) and is denoted as

$f'(x) = \dfrac{dy}{dx} = \overset{lim}{\Delta x \rightarrow 0} \dfrac{f(x+\Delta x)-f(x)}{\Delta x}$

3.6. Geometrical meaning of Differentiation

The geometrical meaning of differentiation is very much useful in the analysis of graphs in physics. To understand the geometrical meaning of derivatives we should have knowledge of secant and tangent to a curve.

Secant and tangent to a curve

Secant : A secant to a curve is a straight line, which intersects the curve at any two points.
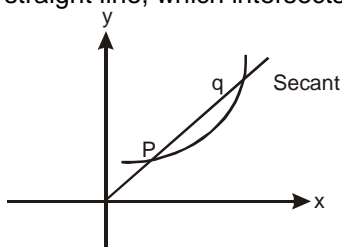
Tangent : A tangent is a straight line, which touches the curve at a particular point. Tangent is a limiting case of secant which intersects the curve at two overlapping points.

In the figure-1 shown, if value of Δx is gradually reduced then the point Q will move nearer to the point P. If the process is continuously repeated (Figure - 2) value of Δx will be infinitely small and secant PQ to the given curve will become a tangent at point P.
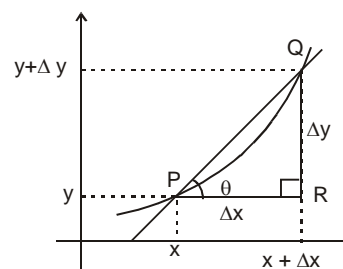
Figure - 1

Therefore $\underset{\Delta x \rightarrow 0}{}\left(\dfrac{\Delta y}{\Delta x}\right) = \dfrac{dy}{dx} = \tan \theta$

we can say that differentiation of y with respect to x, i.e., $\left(\dfrac{dy}{dx}\right)$ is equal to slope of the tangent at point

P (x, y) or $\tan\theta = \dfrac{dy}{dx}$ (From fig. 1, the average rate of change of y from x to x + Δx is identical with the slope of secant PQ.)
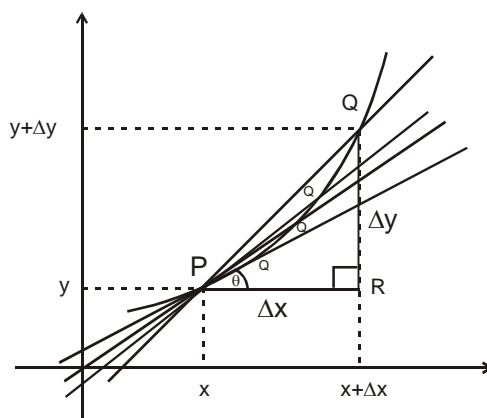

Figure - 2

3.7.   Rules for Differentiation
RULE NO. 1 : DERIVATIVE OF A CONSTANT

The first rule of differentiation is that the derivative of every constant function is zero.

If c is constant, then $\dfrac{d}{dx}c = 0$.

Example 1. $\dfrac{d}{dx}(8) = 0$, $\dfrac{d}{dx}\left(-\dfrac{1}{2}\right) = 0$, $\dfrac{d}{dx}\left(\sqrt{3}\right) = 0$

RULE NO. 2 : POWER RULE

If n is a real number, then $\dfrac{d}{dx}x^n = nx^{n-1}$.

To apply the power Rule, we subtract 1 from the original exponent (n) and multiply the result by n.

Example 1.

f	x	x^2	x^3	x^4	....
f'	1	2x	$3x^2$	$4x^3$	....

Example 2.   (i) $\dfrac{d}{dx}\left(\dfrac{1}{x}\right) = \dfrac{d}{dx}(x^{-1}) = (-1)x^{-2} = -\dfrac{1}{x^2}$

(ii) $\dfrac{d}{dx}\left(\dfrac{4}{x^3}\right) = 4\dfrac{d}{dx}(x^{-3}) = 4(-3)x^{-4} = -\dfrac{12}{x^4}$.

Example 3. (a) $\dfrac{d}{dx}(x^{1/2}) = \dfrac{1}{2}x^{-1/2} = \dfrac{1}{2\sqrt{x}}$

          | |
  Function defined for $x \geq 0$ derivative defined only for $x > 0$

(b) $\dfrac{d}{dx}(x^{1/5})$ $= \dfrac{1}{5}x^{-4/5}$

    | |
Function defined for $x \geq 0$ derivative not defined at $x = 0$

RULE NO. 3 : THE CONSTANT MULTIPLE RULE

If u is a differentiable function of x, and c is a constant, then $\dfrac{d}{dx}(cu) = c\dfrac{du}{dx}$

In particular, if n is a positive integer, then $\dfrac{d}{dx}(cx^n) = cn\,x^{n-1}$

Example 1.   The derivative formula $\dfrac{d}{dx}(3x^2) = 3(2x) = 6x$ says that if we rescale the graph of $y = x^2$ by multiplying each y–coordinate by 3, then we multiply the slope at each point by 3.

Example 2.   A useful special case. The derivative of the negative of a differentiable function is the negative of the function's derivative. Rule 3 with c = –1 gives.

$\dfrac{d}{dx}(-u) = \dfrac{d}{dx}(-1\cdot u) = -1 \cdot \dfrac{d}{dx}(u) = -\dfrac{d}{dx}(u)$

RULE NO. 4 : THE SUM RULE

The derivative of the sum of two differentiable functions is the sum of their derivatives.

If u and v are differentiable functions of x, then their sum u + v is differentiable at every point where u and v are both differentiable functions is their derivatives.

$$\frac{d}{dx}(u-v) = \frac{d}{dx}[u+(-1)v] = \frac{du}{dx}+(-1)\frac{dv}{dx} = \frac{du}{dx}-\frac{dv}{dx}$$

The Sum Rule also extends to sums of more than two functions, as long as there are only finitely many functions in the sum. If $u_1, u_2,.........u_n$ are differentiable at x, then so is

$u_1 + u_2 + ...+ u_n$, and $\frac{d}{dx}(u_1 + u_2 + ... + u_n) = \frac{du_1}{dx} + \frac{du_2}{dx} + ... + \frac{du_n}{dx}$.

Example 3.   **(a)** $y = x^4 + 12x$ **(b)** $y = x^3 + \frac{4}{3}x^2 - 5x + 1$

$$\frac{dy}{dx} = \frac{d}{dx}(x^4) + \frac{d}{dx}(12x) \qquad \frac{dy}{dx} = \frac{d}{dx}\left(x^3\right) + \frac{d}{dx}\left(\frac{4}{3}x^2\right) - \frac{d}{dx}(5x) + \frac{d}{dx}(1)$$

$$= 4x^3 + 12 \qquad\qquad = 3x^2 + \frac{4}{3}.2x - 5 + 0 = 3x^2 + \frac{8}{3}x - 5.$$

Notice that we can differentiate any polynomial term by term, the way we differentiated the polynomials in above example.

RULE NO. 5 : THE PRODUCT RULE

If u and v are differentiable at x, then so is their product uv, and $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$.

The derivative of the product uv is u times the derivative of v plus v times the derivative of u.
In prime notation $(uv)' = uv' + vu'$.
While the derivative of the sum of two functions is the sum of their derivatives, the derivative of the product of two functions is not the product of their derivatives. For instance,

$$\frac{d}{dx}(x.x) = \frac{d}{dx}(x^2) = 2x, \quad \text{while} \quad \frac{d}{dx}(x).\frac{d}{dx}(x) = 1.1 = 1.$$

Example 1.   Find the derivatives of $y = \frac{4}{3}(x^2 + 1)(x^3 + 3)$.

Solution :   From the product Rule with $u = x^2 + 1$ and $v = x^3 + 3$, we find

$$\frac{d}{dx}[(x^2 + 1)(x^3 + 3)] = (x^2 + 1)(3x^2) + (x^3 + 3)(2x)$$

$= 3x^4 + 3x^2 + 2x^4 + 6x = 5x^4 + 3x^2 + 6x$.
Example can be done as well (perhaps better) by multiplying out the original expression for y and differentiating the resulting polynomial. We now check : $y = (x^2 + 1)(x^3 + 3) = x^5 + x^3 + 3x^2 + 3$

$$\frac{dy}{dx} = 5x^4 + 3x^2 + 6x.$$

This is in agreement with our first calculation. There are times, however, when the product Rule must be used. In the following examples. We have only numerical values to work with.

Example 2.   Let y = uv be the product of the functions u and v. Find y'(2) if u(2) = 3, u'(2) = –4, v(2) = 1, and v'(2) = 2.

Solution :   From the Product Rule, in the form $y' = (uv)' = uv' + vu'$,
we have $y'(2) = u(2)v'(2) + v(2)u'(2) = (3)(2) + (1)(-4) = 6 - 4 = 2$.

RULE NO. 6 : THE QUOTIENT RULE

If u and v are differentiable at x, and $v(x) \neq 0$, then the quotient u/v is differentiable at x,

and $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

Just as the derivative of the product of two differentiable functions is not the product of their derivatives, the derivative of the quotient of two functions is not the quotient of their derivatives.

Example 1. Find the derivative of $y = \dfrac{t^2 - 1}{t^2 + 1}$

Solution : We apply the Quotient Rule with $u = t^2 - 1$ and $v = t^2 + 1$:

$$\frac{dy}{dt} = \frac{(t^2+1).2t - (t^2-1).2t}{(t^2+1)^2} \Rightarrow \frac{d}{dt}\left(\frac{u}{v}\right) = \frac{v(du/dt) - u(dv/dt)}{v^2} = \frac{2t^3 + 2t - 2t^3 + 2t}{(t^2+1)^2} = \frac{4t}{(t^2+1)^2}.$$

RULE NO. 7 : DERIVATIVE OF SINE FUNCTION

$$\frac{d}{dx}(\sin x) = \cos x$$

Example 1. (a) $y = x^2 - \sin x$: $\quad \dfrac{dy}{dx} = 2x - \dfrac{d}{dx}(\sin x)$ Difference Rule

$= 2x - \cos x$

(b) $y = x^2 \sin x$: $\dfrac{dy}{dx} = x^2 \dfrac{d}{dx}(\sin x) + 2x \sin x$ Product Rule

$= x^2 \cos x + 2x \sin x$

(c) $y = \dfrac{\sin x}{x}$: $\quad \dfrac{dy}{dx} = \dfrac{x.\dfrac{d}{dx}(\sin x) - \sin x.1}{x^2}$ Quotient Rule

$= \dfrac{x\cos x - \sin x}{x^2}$.

RULE NO. 8 : DERIVATIVE OF COSINE FUNCTION

$$\frac{d}{dx}(\cos x) = -\sin x$$

Example 1. (a) $y = 5x + \cos x$

$\dfrac{dy}{dx} = \dfrac{d}{dx}(5x) + \dfrac{d}{dx}(\cos x)$ Sum Rule

$= 5 - \sin x$

(b) $y = \sin x \cos x$

$\dfrac{dy}{dx} = \sin x \dfrac{d}{dx}(\cos x) + \cos x(\sin x)\dfrac{d}{dx}$ Product Rule

$= \sin x (-\sin x) + \cos x (\cos x) = \cos^2 x - \sin^2 x$

RULE NO. 9 : DERIVATIVES OF OTHER TRIGONOMETRIC FUNCTIONS

Because sin x and cos x are differentiable functions of x, the related functions

$$\tan x = \frac{\sin x}{\cos x} \; ; \qquad \sec x = \frac{1}{\cos x}$$

$$\cot x = \frac{\cos x}{\sin x} \quad ; \qquad \operatorname{cosec} x = \frac{1}{\sin x}$$

are differentiable at every value of x at which they are defined. There derivatives. Calculated from the Quotient Rule, are given by the following formulas.

$$\frac{d}{dx}(\tan x) = \sec^2 x \; ; \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x \; ; \qquad \frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$$

Example 1.   Find dy / dx if y = tan x.

Solution : $\dfrac{d}{dx}(\tan x) = \dfrac{d}{dx}\left(\dfrac{\sin x}{\cos x}\right) = \dfrac{\cos x \dfrac{d}{dx}(\sin x) - \sin x \dfrac{d}{dx}(\cos x)}{\cos^2 x}$

$= \dfrac{\cos x \cos x - \sin x (-\sin x)}{\cos^2 x} = \dfrac{\cos^2 x + \sin^2 x}{\cos^2 x} = \dfrac{1}{\cos^2 x} = \sec^2 x$

Example 2.   (a) $\dfrac{d}{dx}(3x + \cot x) = 3 + \dfrac{d}{dx}(\cot x) = 3 - \operatorname{cosec}^2 x$

(b) $\dfrac{d}{dx}\left(\dfrac{2}{\sin x}\right) = \dfrac{d}{dx}(2\operatorname{cosec} x) = 2\dfrac{d}{dx}(\operatorname{cosec} x) = 2(-\operatorname{cosec} x \cot x) = -2 \operatorname{cosec} x \cot x$

RULE NO. 10 : DERIVATIVE OF LOGARITHM AND EXPONENTIAL FUNCTIONS

$$\frac{d}{dx}(\log_e x) = \frac{1}{x} \quad \Rightarrow \quad \frac{d}{dx}(e^x) = e^x$$

Example 1. y = e^x . log_e (x)

$$\frac{dy}{dx} = \frac{d}{dx}(e^x) \cdot \log(x) + \frac{d}{dx}[\log_e(x)] e^x \quad \Rightarrow \quad \frac{dy}{dx} = e^x \cdot \log_e(x) + \frac{e^x}{x}$$

RULE NO. 11 : CHAIN RULE OR "OUTSIDE INSIDE" RULE

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

It sometimes helps to think about the Chain Rule the following way. If y = f(g(x)),

$$\frac{dy}{dx} = f'[g(x)].g'(x) .$$

In words : To find dy/dx , differentiate the "outside" function f and leave the "inside" g(x) alone ; then multiply by the derivative of the inside.

We now know how to differentiate sin x and $x^2 - 4$, but how do we differentiate a composite like sin $(x^2 - 4)$? The answer is, with the Chain Rule, which says that the derivative of the composite of two differentiable functions is the product of their derivatives evaluated at appropriate points. The Chain Rule is probably the most widely used differentiation rule in mathematics. This section describes the rule and how to use it. We begin with examples.

Solved Example

Example 1. The function y = 6x − 10 = 2(3x − 5) is the composite of the functions y = 2u and u = 3x − 5. How are the derivatives of these three functions related ?

Solution : We have $\dfrac{dy}{dx}$ = 6, $\dfrac{dy}{du}$ = 2, $\dfrac{du}{dx}$ = 3 .

Since 6 = 2.3, $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$

Is it an accident that $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$?

If we think of the derivative as a rate of change, our intuition allows us to see that this relationship is reasonable. For y = f(u) and u = g(x) , if y changes twice as fast as u and u changes three times as fast as x, then we expect y to change six times as fast as x.

Example 2. We sometimes have to use the Chain Rule two or more times to find a derivative. Here is an example. Find the derivative of g(t) = tan (5 − sin 2t)

Solution : g′(t) = $\dfrac{d}{dt}$ (tan(5− sin 2t)

= sec^2 (5− sin 2t). $\dfrac{d}{dt}$ (5 − sin 2t)

Derivative of
tan u with
u = 5 − sin 2t

Derivative of
5 − sin u
with u = 2t

= sec^2 (5 − sin 2t) . (0 − (cos 2t). $\dfrac{d}{dt}$ (2t)

= sec^2 (5 − sin 2t) . (− cos 2t) . 2

= −2(cos 2t) sec^2 (5 − sin 2t)

Example 3. (a) $\dfrac{d}{dx}$ (1 − x^2)$^{1/4}$ = $\dfrac{1}{4}$ (1−x^2)$^{−3/4}$ (−2x)   u = 1 − x^2 and n = 1/4

Function defined
on [−1 , 1]

= $\dfrac{-x}{2(1-x^2)^{3/4}}$

derivative defined
only on (−1 ,1)

(b) $\dfrac{d}{dx}$ sin 2x = cos 2x $\dfrac{d}{dx}$ (2x) = cos 2x . 2 = 2 cos 2x

(c) $\dfrac{d}{dt}$ (A sin (ωt + ϕ)

= A cos (ωt + ϕ) $\dfrac{d}{dt}$ (ωt + ϕ) = A cos (ωt + ϕ). ω. = A ω cos (ωt + ϕ)

RULE NO. 12 : POWER CHAIN RULE

If u(x) is a differentiable function and where n is a Real number, then u^n is differentiable and

$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}, \; \forall n \in R$$

Example 1. (a) $\dfrac{d}{dx}\sin^5 x = 5\sin^4 x \; \dfrac{d}{dx}(\sin x) = 5\sin^4 x \cos x$

(b) $\dfrac{d}{dx}(2x+1)^{-3} = -3(2x+1)^{-4} \; \dfrac{d}{dx}(2x+1) = -3(2x+1)^{-4}(2) = -6(2x+1)^{-4}$

(c) $\dfrac{d}{dx}\left(\dfrac{1}{3x-2}\right) = \dfrac{d}{dx}(3x-2)^{-1} = -1(3x-2)^{-2}(3x-2)\dfrac{d}{dx} = -1(3x-2)^{-2}(3) = -\dfrac{3}{(3x-2)^2}$

In part (c) we could also have found the derivative with the Quotient Rule.

Example 2. Find the value of $\dfrac{d}{dx}(Ax+B)^n$

Solution : Here $u = Ax + B$, $= \dfrac{du}{dx} A$

$\therefore \dfrac{d}{dx}(Ax+B)^n = n(Ax+B)^{n-1} \cdot A$

RULE NO. 13 : RADIAN VS. DEGREES

$$\frac{d}{dx}\sin(x°) = \frac{d}{dx}\sin\left(\frac{\pi x}{180}\right) = \frac{\pi}{180}\cos\left(\frac{\pi x}{180}\right) = \frac{\pi}{180}\cos(x°).$$

3.8. Double differentiation

If f is differentiable function, then its derivative f' is also a function, so f' may have a derivative of its own, denoted by $(f')' = f''$. This new function f'' is called the second derivative of f because it is the derivative of the derivative of f. Using Leibniz notation, we write the second derivative of $y = f(x)$ as

$\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d^2y}{dx^2}$. Another notation is $f''(x) = D_2 f(x) = D^2 f(x)$

INTERPRETATION OF DOUBLE DERIVATIVE

We can interpret $f''(x)$ as the slope of the curve $y = f'(x)$ at the point $(x, f'(x))$. In other words, it is the rate of change of the slope of the original curve $y = f(x)$.

In general, we can interpret a second derivative as a rate of change of a rate of change. The most familiar example of this is acceleration, which we define as follows.

If $s = s(t)$ is the position function of an object that moves in a straight line, we known that its first

derivative represents the velocity v(t) of the object as a function of time : $v(t) = s'(t) = \dfrac{ds}{dt}$

The instantaneous rate of change of velocity with respect to time is called the acceleration a(t) of the object. Thus, the acceleration function is the derivative of the velocity function and is therefore the

second derivative of the position function: $a(t) = v'(t) = s''(t)$ or in Leibniz notation, $a = \dfrac{dv}{dt} = \dfrac{d^2s}{dt^2}$

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Solved Example

Example 1. If $f(x) = x \cos x$, find $f''(x)$.

Solution : Using the Product Rule, we have $f'(x) = x \dfrac{d}{dx}(\cos x) + \cos x \dfrac{d}{dx}(x)$

$= -x \sin x + \cos x$

To find $f''(x)$ we differentiate $f'(x)$: $f''(x) = \dfrac{d}{dx}(-x \sin x + \cos x)$

$= -x \dfrac{d}{dx}(\sin x) + \sin x \dfrac{d}{dx}(-x) + \dfrac{d}{dx}(\cos x) = -x \cos x - \sin x - \sin x = -x \cos x - 2 \sin x$

Example 2. The position of a particle is given by the equation $s = f(t) = t^3 - 6t^2 + 9t$ where t is measured in seconds and s in meters. Find the acceleration at time t. What is the acceleration after 4s ?

Solution : The velocity function is the derivative of the position function : $s = f(t) = t^3 - 6t^2 + 9t$

$\Rightarrow \quad v(t) = \dfrac{ds}{dt} = 3t^2 - 12t + 9$

The acceleration is the derivative of the velocity function: $a(t) = \dfrac{d^2s}{dt^2} = \dfrac{dv}{dt} = 6t - 12$

$\Rightarrow \quad a(4) = 6(4) - 12 = 12 \text{ m/s}^2$

3.9. Application of derivatives

3.9.1 differentiation as a rate of change

$\dfrac{dy}{dx}$ is rate of change of 'y' with respect to 'x' :

For examples :

(i) $v = \dfrac{dx}{dt}$ this means velocity 'v' is rate of change of displacement 'x' with respect to time 't'

(ii) $a = \dfrac{dv}{dt}$ this means acceleration 'a' is rate of change of velocity 'v' with respect to time 't' .

(iii) $F = \dfrac{dp}{dt}$ this means force 'F' is rate of change of momentum 'p' with respect to time 't' .

(iv) $\tau = \dfrac{dL}{dt}$ this means torque 'τ' is rate of change of angular momentum 'L' with respect to time 't'

(v) Power $= \dfrac{dW}{dt}$ this means power 'P' is rate of change of work 'W' with respect to time 't'

(vi) $I = \dfrac{dq}{dt}$ this means current 'I' is rate of flow of charge 'q' with respect to time 't'

Solved Example

Example 1. The area A of a circle is related to its diameter by the equation $A = \dfrac{\pi}{4} D^2$. How fast is the area changing with respect to the diameter when the diameter is 10 m ?

Solution : The (instantaneous) rate of change of the area with respect to the diameter is $\dfrac{dA}{dD} = \dfrac{\pi}{4} 2D = \dfrac{\pi D}{2}$.

When D = 10 m, the area is changing at rate $(\pi/2)$ 10 = 5π m^2/m. This means that a small change ΔD m in the diameter would result in a change of about $5\pi \Delta D m^2$ in the area of the circle.

Example 2. Experimental and theoretical investigations revealed that the distance a body released from rest falls in time t is proportional to the square of the amount of time it has fallen. We express this by saying that
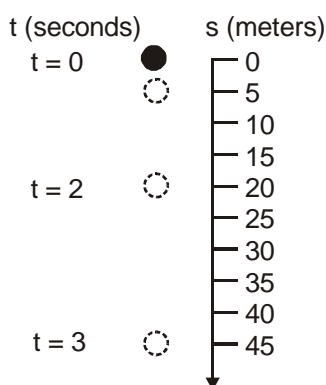
A ball bearing falling from rest

$$s = \dfrac{1}{2} gt^2,$$

where s is distance and g is the acceleration due to Earth's gravity. This equation holds in a vacuum, where there is no air resistance, but it closely models the fall of dense, heavy objects in air. Figure shows the free fall of a heavy ball bearing released from rest at time t = 0 sec.

(a) How many meters does the ball fall in the first 2 sec?

(b) What is its velocity, speed, and acceleration then?

Solution :

(a) The free–fall equation is $s = 4.9 t^2$. During the first 2 sec. the ball falls $s(2) = 4.9(2)^2 = 19.6$ m,

(b) At any time t, velocity is derivative of displacement : $v(t) = s'(t) = \dfrac{d}{dt} (4.9t^2) = 9.8$ t.

At t = 2, the velocity is v(2) = 19.6 m/sec in the downward (increasing s) direction.

The speed at t = 2 is speed = |v(2)| = 19.6 m/sec.

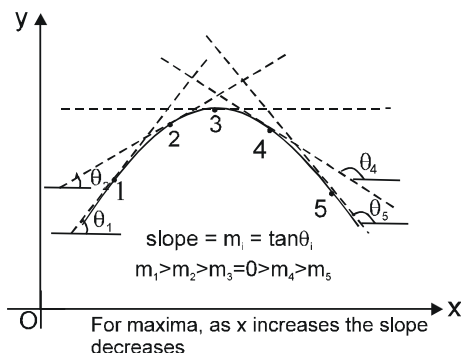$a = \dfrac{d^2s}{dt^2} = 9.8$ m/s^2

3.9.2 Maxima and minima

Suppose a quantity y depends on another quantity x in a manner shown in the figure. It becomes maximum at x_1 and minimum at x_2. At these points the tangent to the curve is parallel to the x-axis and hence its slope is $\tan\theta = 0$. Thus, at a maximum or a minimum, slope = $\dfrac{dy}{dx} = 0$.
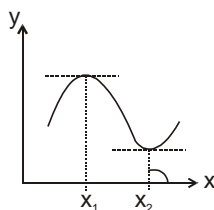
MAXIMA

Just before the maximum the slope is positive, at the maximum it is zero and just after the maximum it is negative. Thus, $\dfrac{dy}{dx}$ decreases at a maximum and hence the rate of change of $\dfrac{dy}{dx}$ is negative at a maximum i.e. $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) < 0$ at maximum.

slope = m_i = $\tan\theta_i$
$m_1 > m_2 > m_3 = 0 > m_4 > m_5$

For maxima, as x increases the slope decreases

The quantity $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$ is the rate of change of the slope. It is written as $\dfrac{d^2y}{dx^2}$.

Conditions for maxima are: (a) $\dfrac{dy}{dx} = 0$ (b) $\dfrac{d^2y}{dx^2} < 0$

MINIMA

Similarly, at a minimum the slope changes from negative to positive. Hence with the increases of x. the slope is increasing that means the rate of change of slope with respect to x is positive hence $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) > 0$.

slope = m_i = $\tan\theta_i$
$m_1 < m_2 < m_3 = 0 < m_4 < m_5$

$\theta_3 = 0$

For minima, as x increases slope increases.
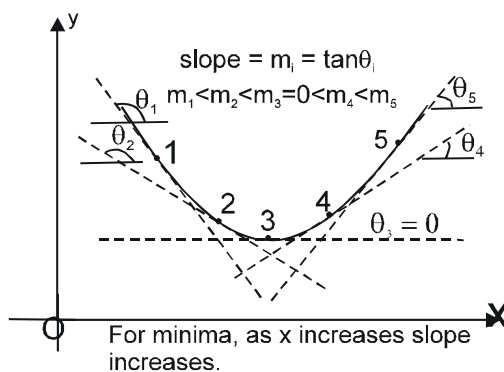
Conditions for minima are : (a) $\dfrac{dy}{dx} = 0$ (b) $\dfrac{d^2y}{dx^2} > 0$

Quite often it is known from the physical situation whether the quantity is a maximum or a minimum. The test on $\dfrac{d^2y}{dx^2}$ may then be omitted.

Solved Example

Example 1. Particle's position as a function of time is given as $x = 5t^2 - 9t + 3$. Find out the maximum value of position co-ordinate? Also, plot the graph.

Solution : $x = 5t^2 - 9t + 3$

$\dfrac{dx}{dt} = 10t - 9 = 0$ $\qquad$ $\therefore$ $t = 9/10 = 0.9$

Check, whether maxima or minima exists. $\dfrac{d^2x}{dt^2} = 10 > 0$

$\therefore$ there exists a minima at $t = 0.9$
Now, Check for the limiting values.
When $t = 0$; $x = 3$
$\qquad$ $t = \infty$; $x = \infty$
So, the maximum position co-ordinate does not exist.

Graph :
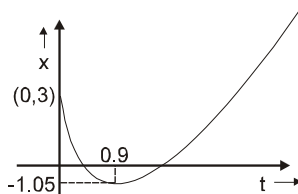
Putting $t = 0.9$ in the equation $x = 5(0.9)^2 - 9(0.9) + 3 = -1.05$

NOTE : If the coefficient of t^2 is positive, the curve will open upside.

3.10. Solved examples on application of derivative

Example 1. Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangents ? If so, where ?

Solution : The horizontal tangents, if any, occur where the slope dy/dx is zero. To find these points. We

1. Calculate dy/dx : $\dfrac{dy}{dx} = \dfrac{d}{dx}(x^4 - 2x^2 + 2) = 4x^3 - 4x$

2. Solve the equation : $\dfrac{dy}{dx} = 0$ for x : $4x^3 - 4x = 0$

$4x(x^2 - 1) = 0$; $x = 0, 1, -1$

The curve $y = x^4 - 2x^2 + 2$ has horizontal tangents at $x = 0, 1$ and -1. The corresponding points on the curve are $(0, 2)$ $(1, 1)$ and $(-1, 1)$. See figure.
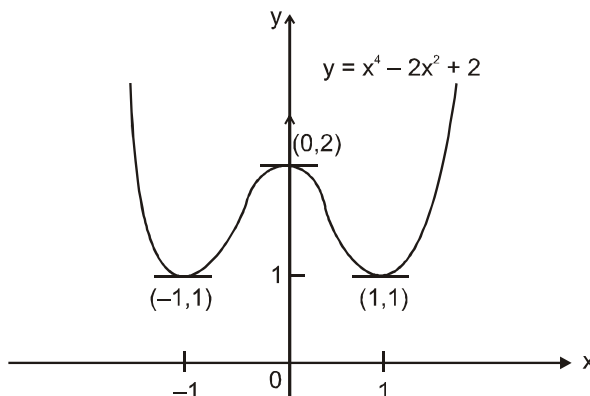
Example 2. A hot air balloon rising straight up from a level field is tracked by a range finder 500 ft from the lift-off point. At the moment the range finder's elevation angle is $\pi/4$, the angle is increasing at the rate of 0.14 rad/min. How fast is the balloon rising at the moment ?

Solution : We answer the question in six steps.

Step-1. Draw a picture and name the variables and constants (Figure). The variables in the picture are

θ = the angle the range finder makes with the ground (radians).

y = the height of the balloon (feet).

We let t represent time and assume θ and y to be differentiable functions of t.

The one constant in the picture is the distance from the range finder to the lift-off point (500 ft.)

There is no need to give it a special symbol s.
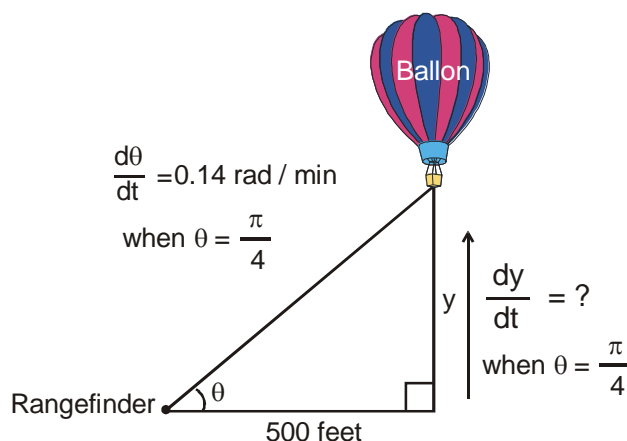
Step-2. Write down the additional numerical information. $\dfrac{d\theta}{dt}$ = 0.14 rad/min when $\theta = \pi / 4$.

Step-3. Write down what we are asked to find. We want dy/dt when $\theta = \pi/4$.

Step-4. Write an equation that relates the variables y and θ. = tan θ, $\dfrac{y}{500}$ or y = 500 tan θ.

Step-5. Differentiate with respect to t using the Chain Rule. The result tells how dy/dt (which we want) is related to dθ/dt (which we know).

$$\frac{dy}{dt} = 500 \sec^2 \theta \ \frac{d\theta}{dt}$$

**Step-6.** Evaluate with $\theta = \pi/4$ and dθ/dt = 0.14 to find dy/dt.

$\dfrac{dy}{dt} = 500\left(\sqrt{2}\right)^2 (0.14) = (1000)\,(0.14)\ = 140 \quad (\sec \dfrac{\pi}{4} = \sqrt{2}\,)$

At the moment in question, the balloon is rising at the rate of 140 ft./min.

Example 3.   A police cruiser, approaching a right-angled intersection from the north, is chasing a speeding car that has turned the corner and is now moving straight east. When the Cruiser is 0.6 mi north of the intersection and the car is 0.8 mi to the east, the police determine with radar that the distance between them and the car is increasing at 20 mph. If the cruiser is moving at 60 mph at the instant of measurement, what is the speed of the car?

Solution :   We carry out the steps of the basic strategy.

Situations when
x=0.8, y=0.6

$\frac{ds}{dt}=20$

$\frac{dy}{dt}=-60$

$\frac{dx}{dt}=?$
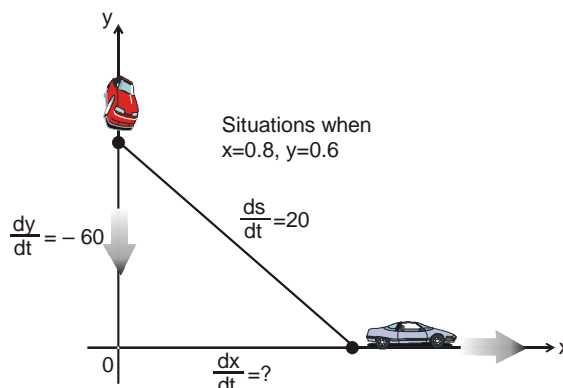
Step-1. Picture and variables. We picture the car and cruiser in the coordinate plane, using the positive x-axis as the eastbound highway and the positive y-axis as the northbound highway (Figure). We let t represent time and set x = position of car at time t.

y = position of cruiser at time t,   s = distance between car and cruiser at time t.

We assume x, y and s to be differentiable functions of t.

$x = 0.8$ mi, $y = 0.6$ mi, $\frac{dy}{dt} = -60$ mph, $\frac{ds}{dt} = 20$ mph

(dy/dt is negative because y is decreasing.)

Step-2. To find : $\frac{dx}{dt}$

Step-3. How the variables are related : $s^2 = x^2 + y^2$ Pythagorean theorem

(The equation $s = \sqrt{x^2 + y^2}$ would also work.)

Step-4. Differentiate with respect to t. $2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$ Chain Rule

$$\frac{ds}{dt} = \frac{1}{s}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right) = \frac{1}{\sqrt{x^2+y^2}}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$

Step-5. Evaluate, with x = 0.8, y = 0.6, dy/dt = − 60, ds/dt = 20, and solve for dx/dt.

$$20 = \underbrace{\frac{1}{\sqrt{(0.8)^2 + (0.6)^2}}}_{1}\left(0.8\frac{dx}{dt} + (0.6)(-60)\right) \Rightarrow 20 = 0.8\frac{dx}{dt} - 36 \Rightarrow \frac{dx}{dt} = \frac{20+36}{0.8} = 70$$

At the moment in question, the car's speed is 70 mph.

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4. INTEGRATION

In mathematics, for each mathematical operation, there has been defined an inverse operation.

For example - Inverse operation of addition is subtraction, inverse operation of multiplication is division and inverse operation of square is square root. Similarly there is a inverse operation for differentiation which is known as integration

4.1. Anti-derivatives OR Indefinite Integrals

Definitions :

A function F(x) is an antiderivative of a function f(x) if F´(x) = f(x) for all x in the domain of f.

The set of all antiderivatives of f is the indefinite integral of f with respect to x, denoted by

The function is the integrand.

x is the variable of integration

Integral sign

$$\int f(x)dx$$

Integral of f

The symbol $\int$ is an integral sign. The function f is the integrand of the integral and x is the variable of integration.

For example $f(x) = x^3$ then $f'(x) = 3x^2$

So the integral of $3x^2$ is x^3

Similarly if $f(x) = x^3 + 4$ then $\qquad f'(x) = 3x^2$

So the integral of $3x^2$ is $x^3 + 4$

there for general integral of $3x^2$ is $x^3 + c$ where c is a constant

One antiderivative F of a function f, the other antiderivatives of f differ from F by a constant. We indicate this in integral notation in the following way :

$$\int f(x)dx = F(x) + C. \qquad\qquad .....(i)$$

The constant C is the constant of integration or arbitrary constant, Equation (1) is read, "The indefinite integral of f with respect to x is F(x) + C." When we find F(x) + C, we say that we have integrated f and evaluated the integral.

Solved Example

Example 1.   Evaluate $\int$ 2x dx.

an antiderivative of 2x

Solution : $\int 2x\,dx = x^2 + C$

the arbitrary constant

The formula $x^2 + C$ generates all the antiderivatives of the function 2x. The function $x^2 + 1$, $x^2 - \pi$,

and $x^2 + \sqrt{2}$ are all antiderivatives of the function 2x, as you can check by differentiation. Many of the indefinite integrals needed in scientific work are found by reversing derivative formulas.

4.2. Integral Formulas

Indefinite Integral	Reversed derivative formula

1. $\int x^n dx = \dfrac{x^{n+1}}{n+1} + C$, $n \neq -1$, n rational $\qquad \dfrac{d}{dx}\left(\dfrac{x^{n+1}}{n+1}\right) = x^n$

2. $\int dx = \int 1 dx = x + C$ (special case) $\qquad \dfrac{d}{dx}(x) = 1$

3. $\int \sin(Ax+B)dx = \dfrac{-\cos(Ax+B)}{A} + C \qquad \dfrac{d}{dx}\left(-\dfrac{\cos kx}{k}\right) = \sin kx$

4. $\int \cos kx\, dx = \dfrac{\sin kx}{k} + C \qquad \dfrac{d}{dx}\left(\dfrac{\sin kx}{k}\right) = \cos kx$

5. $\int \sec^2 x\, dx = \tan x + C \qquad \dfrac{d}{dx}\tan x = \sec^2 x$

6. $\int \mathrm{cosec}^2 x\, dx = -\cot x + C \qquad \dfrac{d}{dx}(-\cot x) = \mathrm{cosec}^2 x$

7. $\int \sec x \tan x\, dx = \sec x + C \qquad \dfrac{d}{dx}\sec x = \sec x \tan x$

8. $\int \mathrm{cosec}\, x \cot x\, dx = -\mathrm{cosec}\, x + C \qquad \dfrac{d}{dx}(-\mathrm{cosec}\, x) = \mathrm{cosec}\, x \cot x$

9. $\int \dfrac{1}{(ax+b)} = \dfrac{1}{a}\ln(ax+b) + C$

Solved Example

Example 1. Examples based on above formulas :

(a) $\int x^5 dx = \dfrac{x^6}{6} + C \qquad\qquad$ Formula 1 with n = 5

(b) $\int \dfrac{1}{\sqrt{x}}dx = \int x^{-1/2}dx = 2x^{1/2} + C = 2\sqrt{x} + C \qquad$ Formula 1 with n = −1/2

(c) $\int \sin 2x\, dx = \dfrac{-\cos 2x}{2} + C \qquad\qquad$ Formula 2 with k = 2

(d) $\int \cos \dfrac{x}{2}dx = \int \cos \dfrac{1}{2}x\, dx = \dfrac{\sin(1/2)x}{1/2} + C = 2\sin \dfrac{x}{2} + C \qquad$ Formula 3 with k = 1/2

Example 2. Right : $\int$ x cosx dx = x sin x + cos x + C

Reason : The derivative of the right-hand side is the integrand:

Check : $\dfrac{d}{dx}$ (x sin x + cos x + C) = x cos x + sin x − sin x + 0 = x cos x.

Wrong : x cosx dx = x sin x + C

Reason : $\int$ The derivative of the right-hand side is not the integrand:

Check : $\dfrac{d}{dx}$ (x sin x + C) = x cos x + sin x + 0 x cos x.

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4.3. Rules for Integration

RULE NO. 1 : CONSTANT MULTIPLE RULE

A function is an antiderivative of a constant multiple kf of a function f if and only if it is k times

an antiderivative of f.

$\int kf(x)dx = k \int f(x)dx$; where k is a constant

Example 1.　Rewriting the constant of integration $\int$　5 sec x tan x d x = 5 $\int$　sec x tan x d x

Rule 1

= 5 (sec x + C)　　　　　　　Formula 6

= 5 sec x + 5C　　　　　　　First form

= 5 sec x + C´　　　　　　　Shorter form, where C' is 5C

= 5 sec x + C　　　　　　　Usual form–no prime. Since 5 times an arbitrary constant is an

　　　　　　　　　　　　　　arbitrary constant, we rename C'.

What about all the different forms in example? Each one gives all the antiderivatives of

f(x) = 5 sec x tan x. so each answer is correct. But the least complicated of the three, and the

usual choice, is

　　　$\int$　5 sec x tan x d x = 5 sec x + C.

Just as the Sum and Difference Rule for differentiation enables us to differentiate expressions

term by term, the Sum and Difference Rule for integration enables us to integrate expressions

term by term. When we do so, we combine the individual constants of integration into a single

arbitrary constant at the end.

RULE NO. 2 : SUM AND DIFFERENCE RULE

A function is an antiderivative of a sum or difference $f \pm g$ if and only if it is the sum or

difference of an antiderivative of f an antiderivative of g.

$\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$

Solved Example

Example 1. Term–by–term integration. Evaluate : $\int (x^2 - 2x + 5)\, dx$.

Solution : If we recognize that $(x^3/3) - x^2 + 5x$ is an antiderivative of $x^2 - 2x + 5$, we can evaluate the integral as

$$(x^2 - 2x + 5)dx = \underbrace{\frac{x^3}{3} - x^2 + 5x}_{\text{antiderivative}} + C \quad \text{arbitrary constant}$$

If we do not recognize the antiderivative right away, we can generate it term by term with the sum and difference Rule:

$$\int (x^2 - 2x + 5)dx = \int x^2 dx - \int 2x\, dx + \int 5\, dx = \frac{x^3}{3} + C_1 - x^2 + C_2 + 5x + C_3.$$

This formula is more complicated than it needs to be. If we combine C_1, C_2 and C_3 into a single constant $C = C_1 + C_2 + C_3$, the formula simplifies to

$$\frac{x^3}{3} - x^2 + 5x + C$$

and still gives all the antiderivatives there are. For this reason we recommend that you go right to the final form even if you elect to integrate term by term. Write

$$\int (x^2 - 2x + 5)dx = \int x^2 dx - \int 2x\, dx + \int 5\, dx = \frac{x^3}{3} - x^2 + 5x + C.$$

Find the simplest antiderivative you can for each part add the constant at the end.

Example 2. We can sometimes use trigonometric identities to transform integrals we do not know how to evaluate into integrals we do know how to evaluate. The integral formulas for $\sin^2 x$ and $\cos^2 x$ arise frequently in applications.

(a) $\int \sin^2 x\, dx = \int \frac{1 - \cos 2x}{2}\, dx \quad \sin^2 x = \frac{1 - \cos 2x}{2}$

$$= \frac{1}{2}\int (1 - \cos 2x)dx = \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x\, dx$$

$$= \frac{x}{2} - \left(\frac{1}{2}\right)\frac{\sin 2x}{2} + C = \frac{x}{2} - \frac{\sin 2x}{4} + C$$

(b) $\int \cos^2 x\, dx = \int \frac{1 + \cos 2x}{2}\, dx \quad \cos^2 x = \frac{1 + \cos 2x}{2}$

$$= \frac{x}{2} + \frac{\sin 2x}{4} + C \quad \text{As in part (a), but with a sign change}$$

Example 3. Find a body velocity from its acceleration and initial velocity. The acceleration of gravity near the surface of the earth is 9.8 m/sec^2. This means that the velocity v of a body falling freely in a vacuum changes at the rate of $\dfrac{dv}{dt}$ = 9.8 m/sec^2. If the body is dropped from rest, what will its velocity be t seconds after it is released?

Solution : In mathematical terms, we want to solve the initial value problem that consists of

The differential condition : $\dfrac{dv}{dt}$ = 9.8

The initial condition : v = 0 when t = 0 (abbreviated as v (0) = 0)

We first solve the differential equation by integrating both sides with respect to t:

$\dfrac{dv}{dt}$ = 9.8   The differential equation

$\int \dfrac{dv}{dt} dt = \int 9.8 dt$ Integrate with respect to t.

v + C$_1$ = 9.8t + C$_2$ Integrals evaluated

v = 9.8t + C.   Constants combined as one

This last equation tells us that the body's velocity t seconds into the fall is 9.8t + C m/sec.

For value of C : What value? We find out from the initial condition :

v = 9.8t + C

0 = 9.8(0) + C   v(0) = 0

C = 0.

Conclusion : The body's velocity t seconds into the fall is

v = 9.8t + 0 = 9.8t m/sec.

The indefinite integral F(x) + C of the function f(x) gives the general solution y = F(x) + C of the differential equation dy/dx = f(x). The general solution gives all the solutions of the equation (there are infinitely many, one for each value of C). We solve the differential equation by finding its general Solution : We then solve the initial value problem by finding the particular solution that satisfies the initial condition y(x$_o$) = y$_o$ (y has the value y$_o$ when x = x$_o$).

RULE NO. 3 : RULE OF SUBSTITUTION

$\int f(g(x)) \cdot g'(x)\, dx = \int f(u) du$ 1. Substitute u = g(x), du = g'(x) dx.

= F(u) + C   2. Evaluate by finding an antiderivative F (u) of f(u). (Any one will do.)

= F (g(x)) + C   3. Replace u by g(x).

Example 1. Evaluate $\int (x + 2)^5 dx$.

We can put the integral in the form $\int u^n du$

by substituting   u = x + 2,   du = d(x + 2) = $\dfrac{d}{dx}$ (x + 2). dx = 1.dx = dx.

Then $\int (x + 2)^5 dx = \int u^5 du$ u = x + 2, du = dx

= $\dfrac{u^6}{6}$ + C   Integrate, using rule no. 3 with n = 5.

= $\dfrac{(x + 2)^6}{6}$ + C.   Replace u by x + 2.

Example 2. Evaluate $\int \sqrt{1+y^2} \cdot 2y\,dy = \int u^{1/2}du$. Let $u = 1 + y^2$, $du = 2y\,dy$.

$$= \frac{u^{(1/2)+1}}{(1/2)+1}$$ Integrate, using rule no. 3 with n = 1/2.

$$= \frac{2}{3}u^{3/2} + C$$ Simpler form

$$= \frac{2}{3}(1+y^2)^{3/2} + C$$ Replace u by $1 + y^2$.

Example 3. Evaluate $\int \cos\ (7\theta + 5)\ d\theta = \int \cos u \cdot \frac{1}{7}du$ Let u = 7θ + 5, du = 7dθ, (1/7) du = dθ.

$$= \frac{1}{7}\int \cos u\,du$$ With (1/7) out front, the integral is now in standard form.

$$= \frac{1}{7}\sin u + C$$ Integrate with respect to u.

$$= \frac{1}{7}\sin\ (7\theta + 5) + C$$ Replace u by 7θ + 5.

Example 4. Evaluate $\int x^2 \sin(x)^3\,dx = \int \sin(x)^3 \cdot x^2\,dx$

$$= \int \sin u \cdot \frac{1}{3}du$$ Let $u = x^3$, $du = 3x^2\,dx$, (1/3) du = x^2dx.

$$= \frac{1}{3}\int \sin u\,du$$

$$= \frac{1}{3}\ (-\cos u) + C$$ Integrate with respect to u.

$$= -\frac{1}{3}\cos(x^3) + C$$ Replace u by x^3.

Example 5. $\int \frac{1}{\cos^2 2\theta}\,d\theta = \int \sec^2\ 2\theta\,d\theta$ $\sec 2\theta = \frac{1}{\cos 2\theta}$

$$= \int \sec^2 u \cdot \frac{1}{2}du$$ Let u = 2θ, du =2dθ, dθ = (1/2)du.

$$= \frac{1}{2}\int \sec^2 u\,du$$

$$= \frac{1}{2}\tan u + C$$ Integrate, using eq. (4).

$$= \frac{1}{2}\tan 2\theta + C$$ Replace u by 2θ.

Check: $\frac{d}{d\theta}\left(\frac{1}{2}\tan 2\theta + C\right) = \frac{1}{2} \cdot \frac{d}{d\theta}\ (\tan 2\theta) + 0 = \frac{1}{2} \cdot \left(\sec^2 2\theta \cdot \frac{d}{d\theta} 2\theta\right)$ Chain Rule

$$= \frac{1}{2} \cdot \sec^2 2\theta \cdot 2 = \frac{1}{\cos^2 2\theta}\ .$$

Example 6. $\int \sin^4 t \cos t\,dt = \int u^4 du$ Let u = sin t, du = cos t dt.

$$= \frac{u^5}{5} + C$$ Integrate with respect to u.
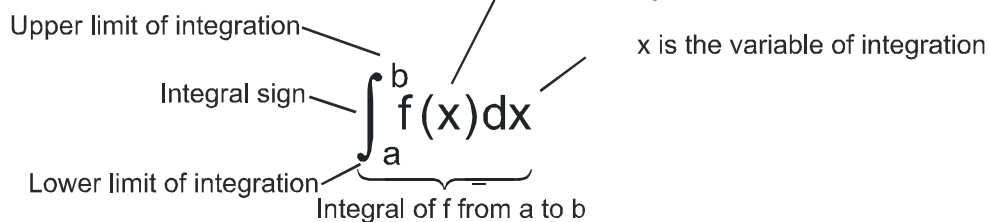
$$= \frac{\sin^5 t}{5} + C$$ Replace u.

The success of the substitution method depends on finding a substitution that will change an integral we cannot evaluate directly into one that we can. If the first substitution fails, we   can try to simplify the integrand further with an additional substitution or two.

4.4. DEFINITE INTEGRATION OR INTEGRATION WITH LIMITS

The function is the integrand.

Upper limit of integration

Integral sign

$$\int_a^b f(x)dx$$

x is the variable of integration

Lower limit of integration

Integral of f from a to b

$$\int_a^b f(x)dx = \left[g(x)\right]_a^b = g(b) - g(a)$$

where g(x) is the antiderivative of f(x) i.e. g′(x) = f(x)

Solved Example

Example 1. $\int_{-1}^{4} 3dx = 3\int_{-1}^{4} dx = 3\left[x\right]_{-1}^{4} = 3[4 - (-1)] = (3)\ (5) = 15$

$\int_0^{\pi/2} \sin x\,dx = \left[-\cos x\right]_0^{\pi/2} = -\cos\left(\frac{\pi}{2}\right) + \cos(0) = -0 + 1 = 1$

4.5. APPLICATION OF DEFINITE INTEGRAL : CALCULATION OF AREA OF A CURVE

From graph shown in figure if we divide whole area in infinitely small strips of dx width. We take a strip at x position of dx width. Small area of this strip dA = f(x) dx
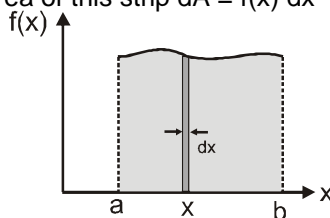
So, the total area between the curve and x-axis

= sum of area of all strips = $\int_a^b f(x)dx$

Let f(x) ≥ 0 be continuous on [a, b]. The area of the region between the graph of f and the x-axis is

$$A = \int_a^b f(x)dx$$

Solved Example

Example 1. Using an area to evaluate a definite integral $\int_a^b x\,dx$ 0 < a < b.

Solution : We sketch the region under the curve y = x, a ≤ x ≤ b (figure) and see that it is a trapezoid with height (b − a) and bases a and b. The value of the integral is the area of this trapezoid :
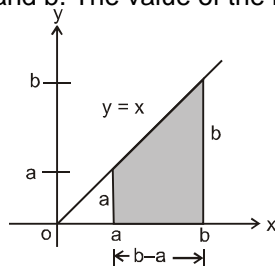
The region in Example

$$\int_a^b x\,dx = (b - a) \cdot \frac{a+b}{2} = \frac{b^2}{2} - \frac{a^2}{2}.$$

Notice that x²/2 is an antiderivative of x, further evidence of a connection between antiderivatives and summation.

Example 2. $I = \int\limits_{0}^{2\pi} \sin(\theta + \phi).d\theta$ where ϕ is a constant. Then value of I :

(A) may be positive   (B) may be negative

(C) may be zero   (D) Always zero for any value of ϕ

Answer : **(D)**

Solution : $\int\limits_{0}^{2n} \sin(\theta + \phi)d\theta$

$= \left| -\cos(\theta + \phi) \right|_{0}^{2\pi}$

$= -[\cos(2n + \phi) - \cos(\phi)]$

$= -[\cos\phi - \cos\phi] = 0$

Example 3. If $x_1 = 3\sin\omega t$ and $x_2 = 4\cos\omega t$ then

(A) $\dfrac{x_1}{x_2}$ is independent of t

(B) Average value of $<x_1^2 + x_2^2>$ from t = 0 to t = $\dfrac{2\pi}{\omega}$ is 12.5

(C) $\left(\dfrac{x_1}{3}\right)^2 + \left(\dfrac{x_2}{4}\right)^2 = 1$

(D) Average value of $<x_1 . x_2>$ from t = 0 to t = $\dfrac{2\pi}{\omega}$ is zero

Answer : **(BCD)**

Solution : (A) $\dfrac{x_1}{x_2} = \dfrac{3\sin\omega t}{4\cos\omega t} = \dfrac{3}{4}\tan\omega t$

                It is dependent of t

(B) $< x_1^2 + x_2^2 > = < 9\sin^2\omega t + 16\cos^2\omega t >$

$$= 9\dfrac{\int\limits_{0}^{2\pi/\omega} \sin^2\omega t\, dt}{\int\limits_{0}^{2\pi/\omega} dt} + 16\dfrac{\int\limits_{0}^{2\pi/\omega} \cos^2\omega t\, dt}{\int\limits_{0}^{2\pi/\omega} dt}$$

$$= 9\dfrac{\int\limits_{0}^{2\pi/\omega} \dfrac{(1-\cos 2\omega t)}{2} dt}{\int\limits_{0}^{2\pi/\omega} dt} + 16\dfrac{\int\limits_{0}^{2\pi/\omega} \dfrac{(1+\cos 2\omega t)}{2} dt}{\int\limits_{0}^{2\pi/\omega} dt}$$

$$= 9 \cdot \dfrac{1}{2} + 16 \cdot \dfrac{1}{2}$$

$$= 12.5 \quad \textbf{Ans.}$$

(C) $\left(\dfrac{x_1}{3}\right)^2 + \left(\dfrac{x_2}{4}\right)^2 = \left(\dfrac{3\sin\omega t}{3}\right)^2 + \left(\dfrac{4\cos\omega t}{4}\right)^2 = \sin^2\omega t + \cos^2\omega t = 1$ **Ans.**

(D) $<x_1 x_2> = < 3\sin\omega t . 4\cos\omega t > = 6 < 2\sin\omega t\cos\omega t >$

$$= 6 < \sin 2\omega t > = 6\dfrac{\int\limits_{0}^{2\pi/\omega} \sin 2\omega t\, dt}{\int\limits_{0}^{2\pi/\omega} dt} = 6 \times 0 = 0 \quad \text{Ans.}$$

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5. VECTOR

In physics we deal with two type of physical quantity one is scalar and other is vector. Each scalar quantities has magnitude.

Magnitude of a physical quantity means product of numerical value and unit of that physical quantity.

For example mass = 4 kg

Magnitude of mass = 4 kg and unit of mass = kg

Example of scalar quantities : mass, speed, distance etc.

Scalar quantities can be added, subtracted and multiplied by simple laws of algebra.

5.1. Definition of Vector

If a physical quantity in addition to magnitude -

(a) has a specified direction.

(b) It should obey commutative law of additions $\vec{A} + \vec{B} = \vec{B} + \vec{A}$

(c) obeys the law of parallelogram of addition, then and then only it is said to be a vector. If any of the above conditions is not satisfied the physical quantity cannot be a vector.

If a physical quantity is a vector it has a direction, but the converse may or may not be true, i.e. if a physical quantity has a direction, it may or may not a be vector. example : pressure, surface tension or current etc. have directions but are not vectors because they do not obey parallelogram law of addition.

The magnitude of a vector ($\vec{A}$) is the absolute value of a vector and is indicated by $|\vec{A}|$ or A.

Example of vector quantity : Displacement, velocity, acceleration, force etc.

Representation of vector :

Geometrically, the vector is represented by a line with an arrow indicating the direction of vector as

Tail ———————→ Head
|←— Length —→|
(magnitude)
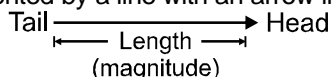
Mathematically, vector is represented by $\vec{A}$

Sometimes it is represented by bold letter **A**.

IMPORTANT POINTS :

☞ If a vector is displaced parallel to itself it does not change (see Figure)

$\vec{A} = \vec{B} = \vec{C}$
Transition of a vector
parallel to itself
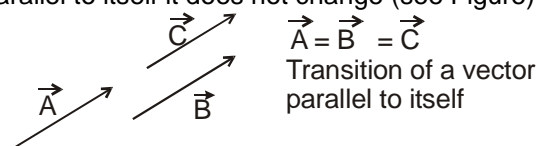
☞ If a vector is rotated through an angle other than multiple

$\vec{A} \neq \vec{B}$ Rotation of a vector
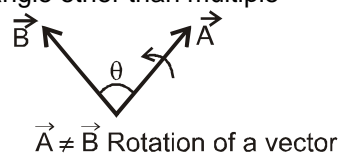
of 2π (or 360º) it changes (see Figure).

☞ If the frame of reference is translated or rotated the vector does not change (though its components may change). (see Figure).

moving frame
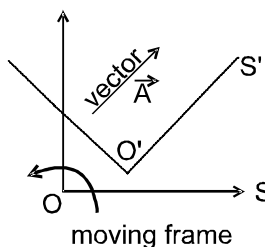
☞ Two vectors are called equal if their magnitudes and directions are same, and they   represent values of same physical quantity.

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☞ Angle between two vectors means smaller of the two angles between the vectors when they are placed tail to tail by displacing either of the vectors parallel to itself (i.e., $0 \le \theta \le \pi$).

5.2.   Unit Vector

Unit vector is a vector which has a unit magnitude and points in a particular direction. Any vector ($\vec{A}$) can be written as the product of unit vector ($\hat{A}$) in that direction and magnitude of the given vector.
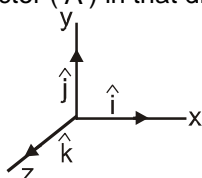
$$\vec{A} = A\hat{A} \quad \text{or} \quad \hat{A} = \frac{\vec{A}}{A}$$

A unit vector has no dimensions and unit. Unit vectors along the positive x-, y- and z-axes of a rectangular coordinate system are denoted by $\hat{i}$, $\hat{j}$ and $\hat{k}$ respectively such that $|\hat{i}| = |\hat{j}| = |\hat{k}| = 1$.
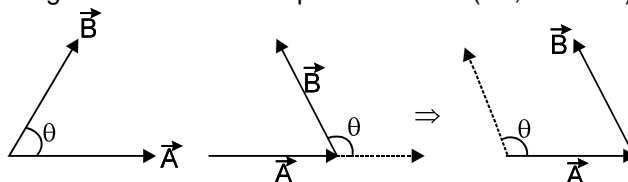
Solved Example

Example 1.   Three vectors $\vec{A}$, $\vec{B}$, $\vec{C}$ are shown in the figure. Find angle between (i) $\vec{A}$ and $\vec{B}$, (ii) $\vec{B}$ and $\vec{C}$, (iii) $\vec{A}$ and $\vec{C}$.
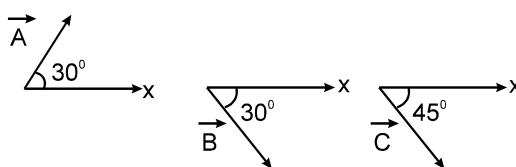
Solution :   To find the angle between two vectors we connect the tails of the two vectors. We can shift $\vec{B}$ such that tails of $\vec{A}$, $\vec{B}$ and $\vec{C}$ are connected as shown in figure.
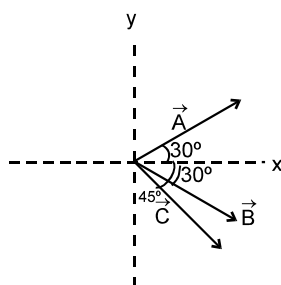
Now we can easily observe that angle between $\vec{A}$ and $\vec{B}$ is 60º, $\vec{B}$ and $\vec{C}$ is 15º and between $\vec{A}$ and $\vec{C}$ is 75º.

Example 2.   A unit vector along East is defined as $\hat{i}$. A force of 10^5 dynes acts west wards. Represent the force in terms of $\hat{i}$.

Solution : $\vec{F} = -10^5 \hat{i}$ dynes

5.3. Multiplication of a Vector by a Scalar

Multiplying a vector $\vec{A}$ with a positive number λ gives a vector $\vec{B}\,(=\lambda\,\vec{A})$ whose magnitude is changed by the factor λ but the direction is the same as that of $\vec{A}$. Multiplying a vector $\vec{A}$ by a negative number λ gives a vector $\vec{B}$ whose direction is opposite to the direction of $\vec{A}$ and whose magnitude is $-\lambda$ times.

Solved Example

Example 1. A physical quantity (m = 3kg) is multiplied by a vector $\vec{a}$ such that $\vec{F}=m\vec{a}$. Find the magnitude and direction of $\vec{F}$ if

(i) $\vec{a}$ = 3m/s² East wards

(ii) $\vec{a}$ = –4m/s² North wards

Solution :

(i) $\vec{F}=m\vec{a}$ = 3 × 3 ms⁻² East wards = 9 N East wards

(ii) $\vec{F}=m\vec{a}$ = 3 × (–4) N North wards = – 12N North wards = 12 N South wards

5.4. Addition of Vectors

Addition of vectors is done by parallelogram law or the triangle law :

(a) Parallelogram law of addition of vectors : If two vectors $\vec{A}$ and $\vec{B}$ are represented by two adjacent sides of a parallelogram both pointing outwards (and their tails coinciding) as shown. Then the diagonal drawn through the intersection of the two vectors represents the resultant (i.e., vector sum of $\vec{A}$ and $\vec{B}$).

$R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$

The direction of resultant vector from $\vec{R}$ is $\vec{A}$ given by

$\tan\phi = \dfrac{MN}{PN} = \dfrac{MN}{PQ+QN} = \dfrac{B\sin\theta}{A+B\cos\theta}$

$\phi = \tan^{-1}\left(\dfrac{B\sin\theta}{A+B\cos\theta}\right)$
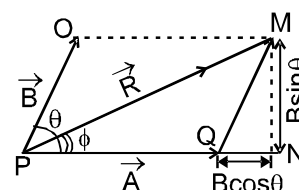
(b) Triangle law of addition of vectors: To add two vectors $\vec{A}$ and $\vec{B}$ shift any of the two vectors parallel to itself until the tail of $\vec{B}$ is at the head of $\vec{A}$. The sum $\vec{A}+\vec{B}$ is a vector $\vec{R}$ drawn from the tail of $\vec{A}$ to the head of $\vec{B}$, i.e., $\vec{A}+\vec{B}=\vec{R}$. As the figure formed is a triangle, this method is called 'triangle method' of addition of vectors.

If the 'triangle method' is extended to add any number of vectors in one operation as shown. Then the figure formed is a polygon and hence the name Polygon Law of addition of vectors is given to such type of addition.

$\vec{A}+\vec{B}=\vec{R}$ $\vec{R}=\vec{A}+\vec{B}+\vec{C}+\vec{D}+\vec{E}$
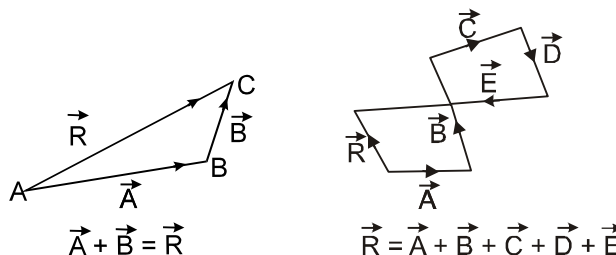
IMPORTANT POINTS :

☞ To a vector only a vector of same type can be added that represents the same physical quantity and the resultant is a vector of the same type.

☞ As R = $[A^2 + B^2 + 2AB \cos\theta]^{1/2}$ so R will be maximum when, $\cos\theta$ = max = 1, i.e., $\theta = 0^0$, i.e., vectors are like or parallel and R_{max} = A + B.

☞ The resultant will be minimum if, $\cos\theta$ = min = -1, i.e., $\theta = 180^0$, i.e., vectors are antiparallel and R_{min} = A ~ B.

☞ If the vectors A and B are orthogonal, i.e., $\theta = 90^0$, R = $\sqrt{A^2 + B^2}$

☞ As previously mentioned that the resultant of two vectors can have any value from (A ~ B) to (A+B) depending on the angle between them and the magnitude of resultant decreases as θ increases 0^0 to 180^0

☞ Minimum number of unequal coplanar vectors whose sum can be zero is three.

☞ The resultant of three non-coplanar vectors can never be zero, or minimum number of non-coplanar vectors whose sum can be zero is four.

☞ Subtraction of a vector from a vector is the addition of negative vector, i.e.,

$\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$

(a) From figure it is clear $\vec{A} - \vec{B}$ that is equal to addition of $\vec{A}$ with reverse of $\vec{B}$
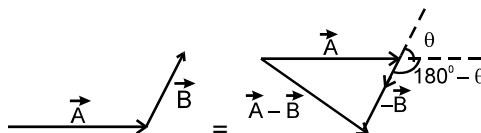
$|\vec{A} - \vec{B}| = [(A)^2 + (B^2) + 2AB \cos(180^0 - \theta)]^{1/2}$

$|\vec{A} - \vec{B}| = \sqrt{A^2 + B^2 - 2AB\cos\theta}$

(b) Change in a vector physical quantity means subtraction of initial vector from the final vector.
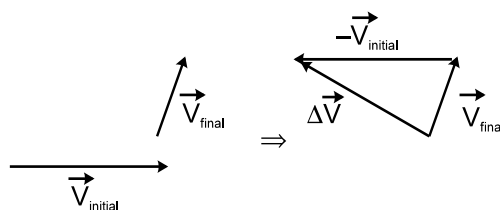
Solved Example

Example 1. Find the resultant of two forces each having magnitude F_0, and angle between them is θ.

Solution : $F^2_{Resultant} = F_0^2 + F_0^2 + 2F_0^2 \cos\theta$

$= 2F_0^2 (1 + \cos\theta) = 2F_0^2 (1 + 2\cos^2\frac{\theta}{2} - 1) = 2F_0^2 \times 2\cos^2\frac{\theta}{2}$

$F_{resultant} = 2F_0 \cos\frac{\theta}{2}$

Example 2. Two non zero vectors $\vec{A}$ and $\vec{B}$ are such that $|\vec{A} + \vec{B}| = |\vec{A} - \vec{B}|$. Find angle between $\vec{A}$ and $\vec{B}$?

Solution : $|\vec{A} + \vec{B}| = |\vec{A} - \vec{B}| \quad \Rightarrow \quad A^2 + B^2 + 2AB \cos\theta = A^2 + B^2 - 2AB \cos\theta$

$\Rightarrow \quad 4AB \cos\theta = 0 \quad \Rightarrow \quad \cos\theta = 0 \quad \Rightarrow \quad \theta = \dfrac{\pi}{2}$

Example 3. The resultant of two velocity vectors $\vec{A}$ and $\vec{B}$ is perpendicular to $\vec{A}$. Magnitude of Resultant is $\vec{R}$ equal to half magnitude of $\vec{B}$. Find the angle between $\vec{A}$ and $\vec{B}$?

Solution : Since $\vec{R}$ is perpendicular to $\vec{A}$. Figure shows the three vectors $\vec{A}$, $\vec{B}$ and $\vec{R}$. angle between $\vec{A}$ and $\vec{B}$ is $\pi - \theta$

$\sin\theta = \dfrac{R}{B} = \dfrac{B}{2B} = \dfrac{1}{2} \Rightarrow \theta = 30^{\circ}$

$\Rightarrow$ angle between A and B is 150°.
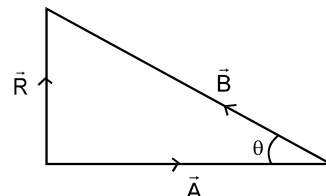
Example 4. If the sum of two unit vectors is also a unit vector. Find the magnitude of their difference?

Solution : Let $\hat{A}$ and $\hat{B}$ are the given unit vectors and $\hat{R}$ is their resultant then $|\hat{R}| = |\hat{A} + \hat{B}|$

$1 = \sqrt{(\hat{A})^2 + (\hat{B})^2 + 2|\hat{A}||\hat{B}|\cos\theta}$

$1 = 1 + 1 + 2\cos\theta \Rightarrow \cos\theta = -\dfrac{1}{2}$

$|\vec{A} - \vec{B}| = \sqrt{(\hat{A})^2 + (\hat{B})^2 - 2|\hat{A}||\hat{B}|\cos\theta} = \sqrt{1 + 1 - 2\times1\times1(-\dfrac{1}{2})} = \sqrt{3}$

5.5. Resolution of Vectors

If $\vec{a}$ and $\vec{b}$ be any two nonzero vectors in a plane with different directions and $\vec{A}$ be another vector in the same plane. $\vec{A}$ can be expressed as a sum of two vectors - one obtained by multiplying $\vec{a}$ by a real number and the other obtained by multiplying $\vec{b}$ by another real number.

$\vec{A} = \lambda\vec{a} + \mu\vec{b}$ (where λ and μ are real numbers)

We say that $\vec{A}$ has been resolved into two component vectors namely $\lambda\vec{a}$ and $\mu\vec{b}$

$\lambda\vec{a}$ and $\mu\vec{b}$ along $\vec{a}$ and $\vec{b}$ respectively. Hence one can resolve a given vector into two component vectors along a set of two vectors − all the three lie in the same plane.

Resolution along rectangular component :

It is convenient to resolve a general vector along axes of a rectangular coordinate system using vectors of unit magnitude, which we call as unit vectors. $\hat{i}, \hat{j}, \hat{k}$ are unit vector along x, y and z-axis as shown in figure below:
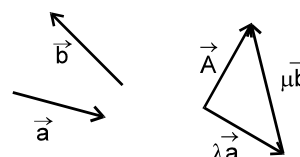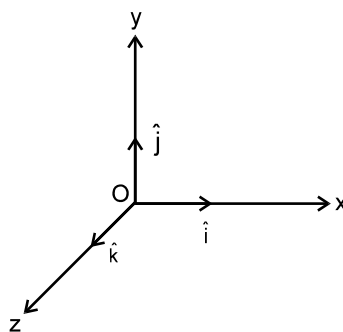
Resolution in two Dimension

Consider a vector $\vec{A}$ that lies in xy plane as shown in figure,

$$\vec{A} = \vec{A}_1 + \vec{A}_2$$

$$\vec{A}_1 = A_x\hat{i}, \ \vec{A}_2 = A_y\hat{j}$$

$$\Rightarrow \ \vec{A} = A_x\hat{i} + A_y\hat{j}$$

The quantities A_x and A_y are called x- and y-components of the vector.

A_x is itself not a vector but $A_x\hat{i}$ is a vector and so is $A_y\hat{j}$
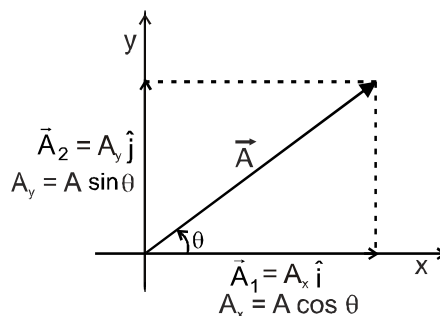
$$A_x = A \cos \theta \text{ and } A_y = A \sin \theta$$

Its clear from above equation that a component of a vector can be positive, negative or zero depending on the value of θ. A vector $\vec{A}$ can be specified in a plane by two ways :

(a) its magnitude A and the direction θ it makes with the x-axis; or

(b) its components A_x and A_y. $\quad A = \sqrt{A_x^2 + A_y^2}$, $\theta = \tan^{-1} \dfrac{A_y}{A_x}$

Note : If $A = A_x \Rightarrow A_y = 0$ and if $A = A_y \Rightarrow A_x = 0$ i.e., components of a vector perpendicular to itself is always zero.

The rectangular components of each vector and those of the sum $\vec{C} = \vec{A} + \vec{B}$ are shown in figure. We saw that

$\vec{C} = \vec{A} + \vec{B}$ is equivalent to both $C_x = A_x + B_x$ and $C_y = A_y + B_y$

Resolution in three dimensions. A vector $\vec{A}$ in components along x-, y- and z-axis can be written as :

$$\overrightarrow{OP} = \overrightarrow{OB} + \overrightarrow{BP} = \overrightarrow{OC} + \overrightarrow{CB} + \overrightarrow{BP}$$

$$\Rightarrow \ \vec{A} = \vec{A}_z + \vec{A}_x + \vec{A}_y = \vec{A}_x + \vec{A}_y + \vec{A}_z = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

$A_x = A \cos \alpha, \ A_y = A \cos \beta, \ A_z = A \cos \gamma$

where $\cos \alpha$, $\cos \beta$ and $\cos \gamma$ are termed as **Direction Cosines** of a

given vector $\vec{A}$.$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$

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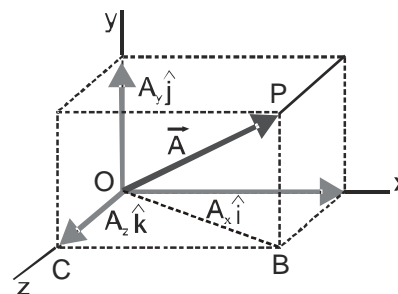
Solved Example

Example 1. A mass of 2 kg lies on an inclined plane as shown in figure. Resolve its weight along and perpendicular to the plane.(Assume g = 10m/s^2)
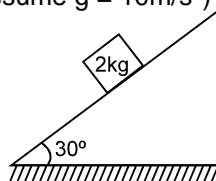
Solution : Component along the plane
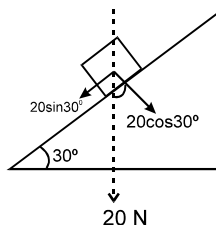
= 20 sin 30 = 10 $\sqrt{3}$ N.

Component perpendicular to the plane = 20 cos 30 = 10N

Example 2. A vector makes an angle of 30º with the horizontal. If horizontal component of the vector is 250. Find magnitude of vector and its vertical component ?
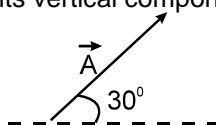
Solution : Let vector is $\overset{\text{\tiny l}}{A}$

$A_x = A \cos 30^0 = 250 = \dfrac{A\sqrt{3}}{2}$ $\Rightarrow$ $A = \dfrac{500}{\sqrt{3}}$

$A_y = A \sin 30^0 = \dfrac{500}{\sqrt{3}} \times \dfrac{1}{2} = \dfrac{250}{\sqrt{3}}$
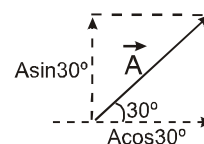
Example 3. . $\overset{\text{\tiny l}}{A} = \hat{i} + 2\hat{j} - 3\hat{k}$, when a vector $\overset{\text{\tiny l}}{B}$ is added to $\overset{\text{\tiny l}}{A}$, we get a unit vector along x-axis. Find the value of $\overset{\text{\tiny l}}{B}$? Also find its magnitude

Solution : $\overset{\text{\tiny r}}{A} + \overset{\text{\tiny r}}{B} = \hat{i}$

$\overset{\text{\tiny l}}{B} = \hat{i} - \overset{\text{\tiny l}}{A} = \hat{i} - (\hat{i} + 2\hat{j} - 3\hat{k}) = -2\hat{j} + 3\hat{k} \Rightarrow |\overset{\text{\tiny l}}{B}| = \sqrt{(2)^2 + (3)^2} = \sqrt{13}$

Example 4. In the above question find a unit vector along $\overset{\text{\tiny l}}{B}$?

Solution : $\hat{B} = \dfrac{\overset{\text{\tiny l}}{B}}{B} = \dfrac{-2\hat{j} + 3\hat{k}}{\sqrt{13}}$

Example 5. Vector $\overset{\text{\tiny l}}{A}$, $\overset{\text{\tiny l}}{B}$ and $\overset{\text{\tiny l}}{C}$ have magnitude 5, 5$\sqrt{2}$ and 5 respectively, direction of $\overset{\text{\tiny l}}{A}$, $\overset{\text{\tiny l}}{B}$ and $\overset{\text{\tiny l}}{C}$ are towards east, North-East and North respectively. If $\hat{i}$ and $\hat{j}$ are unit vectors along East and North respectively. Express the sum $\overset{\text{\tiny l}}{A} + \overset{\text{\tiny l}}{B} + \overset{\text{\tiny l}}{C}$ in terms of $\hat{i}$, $\hat{j}$. Also find magnitude and direction of the resultant.

Solution : $\overset{\text{\tiny l}}{A} = 5\hat{i}$ $\overset{\text{\tiny l}}{C} = 5\hat{j}$

$\overset{\text{\tiny l}}{B} = 5\sqrt{2} \cos 45\hat{i} + 5\sqrt{2} \sin 45\hat{j} = 5\hat{i} + 5\hat{j}$

$\overset{\text{\tiny l}}{A} + \overset{\text{\tiny l}}{B} + \overset{\text{\tiny l}}{C} = 5\hat{i} + 5\hat{i} + 5\hat{j} + 5\hat{j} = 10\hat{i} + 10\hat{j}$

$|\overset{\text{\tiny r}}{A} + \overset{\text{\tiny r}}{B} + \overset{\text{\tiny r}}{C}| = \sqrt{(10)^2 + (10)^2} = 10\sqrt{2}$

$\tan \theta = \dfrac{10}{10} = 1$ $\Rightarrow$ $\theta = 45^0$ from East
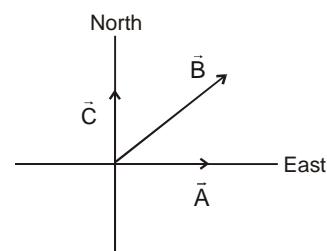
Example 6.   You walk 3 Km west and then 4 Km headed 60° north of east. Find your resultant displacement
        (a) graphically and
        (b) using vector components.

Solution :   Picture the Problem : The triangle formed by the three vectors is not a right triangle, so the magnitudes of the vectors are not related by the Pythagoras theorem. We find the resultant graphically by drawing each of the displacements to scale and measuring the resultant displacement.
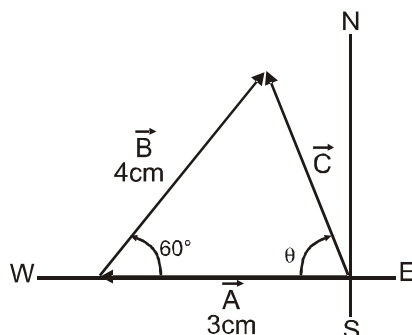
(a) If we draw the first displacement vector 3 cm long and the second one 4 cm long, we find the resultant vector to be about 3.5 cm long. Thus the magnitude of the resultant displacement is 3.5 Km. The angle θ made between the resultant displacement and the west direction can then be measured with a protractor. It is about 75°.

(b)

1. Let $\vec{A}$ be the first displacement and choose the x-axis to be in the easterly direction. Compute A_x and A_y , $A_x = -3$, $A_y = 0$

2. Similarly, compute the components of the second displacement $\vec{B}$, $B_x = 4\cos 60° = 2$, $B_y = 4\sin 60° = 2\sqrt{3}$

3. The components of the resultant displacement $\vec{C} = \vec{A} + \vec{B}$ are found by addition,
$\vec{C} = (-3+2)\hat{i} + (2\sqrt{3})\hat{j} = -\hat{i} + 2\sqrt{3}\,\hat{j}$

4. The Pythagorean theorem gives the magnitude of $\vec{C}$.
$$C = \sqrt{1^2 + \left(2\sqrt{3}\right)^2} = \sqrt{13} = 3.6$$

5. The ratio of C_y to C_x gives the tangent of the angle θ between $\vec{C}$ and the x axis.
$$\tan\theta = \frac{2\sqrt{3}}{-1} \implies \theta = -74°$$

Remark : Since the displacement (which is a vector) was asked for, the answer must include either the magnitude and direction, or both components. in (b) we could have stopped at step 3 because the x and y components completely define the displacement vector. We converted to the magnitude and direction to compare with the answer to part (a). Note that in step 5 of (b), a calculator gives the angle as –74°. But the calculator can't distinguish whether the x or y components is negative. We noted on the figure that the resultant displacement makes an angle of about 75° with the negative x axis and an angle of about 105° with the positive x axis. This agrees with the results in (a) within the accuracy of our measurement.

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5.6. Multiplication of Vectors

5.6.1 The scalar product

The scalar product or dot product of any two vectors $\vec{A}$ and $\vec{B}$, denoted as $\vec{A} \cdot \vec{B}$ (read $\vec{A}$ dot $\vec{B}$) is defined as the product of their magnitude with cosine of angle between them. Thus, $\vec{A} \cdot \vec{B} = AB \cos \theta$ {here θ is the angle between the vectors}

PROPERTIES :

☞ It is always a scalar which is positive if angle between the vectors is acute (i.e. $< 90^\circ$) and negative if angle between them is obtuse (i.e. $90^\circ < \theta \leq 180^\circ$)

☞ It is commutative, i.e., $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$

☞ It is distributive, i.e. $\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$

☞ As by definition $\vec{A} \cdot \vec{B} = AB \cos \theta$. The angle between the vectors $\theta = \cos^{-1} \left[\dfrac{\vec{A} \cdot \vec{B}}{AB} \right]$

☞ $\vec{A} \cdot \vec{B} = A(B \cos \theta) = B(A \cos \theta)$

Geometrically, $B \cos \theta$ is the projection of $\vec{B}$ onto $\vec{A}$ and $A \cos \theta$ is the projection of $\vec{A}$ onto $\vec{B}$ as shown. So $\vec{A} \cdot \vec{B}$ is the product of the magnitude of $\vec{A}$ and the component of $\vec{B}$ along $\vec{A}$ and vice versa.
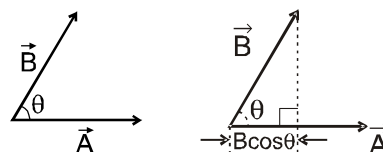
Component of $\vec{B}$ along $\vec{A} = B \cos\theta = \dfrac{\vec{A} \cdot \vec{B}}{A} = \hat{A} \cdot \vec{B}$

Component of $\vec{A}$ along $\vec{B} = A \cos\theta = \dfrac{\vec{A} \cdot \vec{B}}{B} = \vec{A} \cdot \hat{B}$
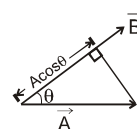
☞ Scalar product of two vectors will be maximum when $\cos \theta = \max = 1$, i.e., $\theta = 0^\circ$, i.e., vectors are parallel $\Rightarrow (\vec{A} \cdot \vec{B})_{max} = AB$

☞ If the scalar product of two nonzero vectors vanishes then the vectors are perpendicular.

☞ The scalar product of a vector by itself is termed as self dot product and is given by

$(\vec{A})^2 = \vec{A} \cdot \vec{A} = AA \cos\theta = AA \cos 0^\circ = A^2 \qquad \Rightarrow \quad A = \sqrt{\vec{A} \cdot \vec{A}}$

☞ In case of unit vector $\hat{n}$,

$\hat{n} \cdot \hat{n} = 1 \times 1 \times \cos 0^\circ = 1 \qquad \Rightarrow \qquad \hat{n} \cdot \hat{n} = \hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$

☞ In case of orthogonal unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$; $\qquad \hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$

$\vec{A} \cdot \vec{B} = (\hat{i} A_x + \hat{j} A_y + \hat{k} A_z) \cdot (\hat{i} B_x + \hat{j} B_y + \hat{k} B_z) = [A_x B_x + A_y B_y + A_z B_z]$

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Solved Example

Example 1. If the Vectors $\vec{P} = a\hat{i} + a\hat{j} + 3\hat{k}$ and $\vec{Q} = a\hat{i} - 2\hat{j} - \hat{k}$ are perpendicular to each other. Find the value of a?

Solution : If vectors $\vec{P}$ and $\vec{Q}$ are perpendicular

$\Rightarrow \vec{P} \cdot \vec{Q} = 0 \quad\quad \Rightarrow (a\hat{i} + a\hat{j} + 3\hat{k}).(a\hat{i} - 2\hat{j} - \hat{k}) = 0 \Rightarrow a^2 - 2a - 3 = 0$

$\Rightarrow a^2 - 3a + a - 3 = 0 \Rightarrow a(a - 3) + 1(a - 3) = 0 \quad\quad \Rightarrow a = -1, 3$

Example 2. Find the component of $3\hat{i} + 4\hat{j}$ along $\hat{i} + \hat{j}$?

Solution : Component of $\vec{A}$ along $\vec{B}$ is given by $\dfrac{\vec{A} \cdot \vec{B}}{B}$ hence required component

$= \dfrac{(3\hat{i} + 4\hat{j}) \cdot (\hat{i} + \hat{j})}{\sqrt{2}} = \dfrac{7}{\sqrt{2}}$

Example 3. Find angle between $\vec{A} = 3\hat{i} + 4\hat{j}$ and $\vec{B} = 12\hat{i} + 5\hat{j}$?

Solution : We have $\cos \theta = \dfrac{\vec{A} \cdot \vec{B}}{AB} = \dfrac{(3\hat{i} + 4\hat{j}) \cdot (12\hat{i} + 5\hat{j})}{\sqrt{3^2 + 4^2}\sqrt{12^2 + 5^2}}$

$\cos \theta = \dfrac{36 + 20}{5 \times 13} = \dfrac{56}{65} \quad\quad \theta = \cos^{-1}\dfrac{56}{65}$

5.7. Vector Product

The vector product or cross product of any two vectors $\vec{A}$ and $\vec{B}$, denoted as $\vec{A} \times \vec{B}$ (read $\vec{A}$ cross $\vec{B}$) is defined as : $\vec{A} \times \vec{B} = AB \sin \theta \, \hat{n}$.

Here θ is the angle between the vectors and the direction $\hat{n}$ is given by the right-hand-thumb rule.

Right-Hand-Thumb Rule:

To find the direction of $\hat{n}$, draw the two vectors $\vec{A}$ and $\vec{B}$ with both the tails coinciding. Now place your stretched right palm perpendicular to the plane of $\vec{A}$ and $\vec{B}$ in such a way that the fingers are along the vector $\vec{A}$ and when the fingers are closed they go towards $\vec{B}$.
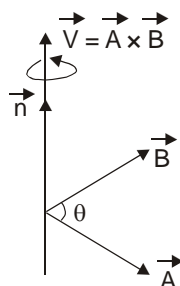
The direction of the thumb gives the direction of $\hat{n}$.

PROPERTIES :

☞ Vector product of two vectors is always a vector perpendicular to the plane containing the two vectors i.e. orthogonal to both the vectors $\vec{A}$ and $\vec{B}$, though the vectors $\vec{A}$ and $\vec{B}$ may or may not be orthogonal.

☞ Vector product of two vectors is not commutative i.e., $\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$.
But $\left|\vec{A} \times \vec{B}\right| = \left|\vec{B} \times \vec{A}\right| = AB \sin \theta$

☞ The vector product is distributive when the order of the vectors is strictly maintained i.e.
$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$.

☞ The magnitude of vector product of two vectors will be maximum when $\sin\theta = \max = 1$, i.e., $\theta = 90^\circ$. $|\vec{A} \times \vec{B}|_{max} = AB$ i.e., magnitude of vector product is maximum if the vectors are orthogonal.

☞ The magnitude of vector product of two non–zero vectors will be minimum when $|\sin\theta| = \text{minimum} = 0$, i.e., $\theta = 0^\circ$ or 180° and $|\vec{A} \times \vec{B}|_{min} = 0$ i.e., if the vector product of two non–zero vectors vanishes, the vectors are collinear.
Note : When $\theta = 0^\circ$ then vectors may be called as like vector or parallel vectors and when $\theta = 180^\circ$ then vectors may be called as unlike vectors or antiparallel vectors.

☞ The self cross product i.e. product of a vector by itself vanishes i.e. is a null vector.
Note : Null vector or zero vector : A vector of zero magnitude is called zero vector. The direction of a zero vector is in determinate (unspecified). $\vec{A} \times \vec{A} = AA \sin 0^\circ \ \hat{n} = \vec{0}$.
In case of unit vector $\hat{n}$, $\hat{n} \times \hat{n} = \vec{0} \Rightarrow \hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$

☞ In case of orthogonal unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ in accordance with right-hand-thumb-rule,
$\hat{i} \times \hat{j} = \hat{k}$ \qquad $\hat{j} \times \hat{k} = \hat{i}$ \qquad $\hat{k} \times \hat{i} = \hat{j}$

(A) \qquad\qquad (B)
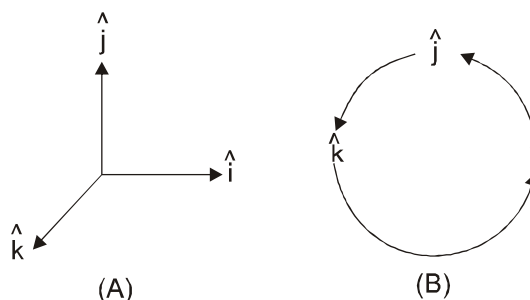
☞ In terms of components,
$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \hat{i}\begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} - \hat{j}\begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} + \hat{k}\begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}$$

$$\vec{A} \times \vec{B} = \hat{i}(A_yB_z - A_zB_y) + \hat{j}(A_zB_x - A_xB_z) + \hat{k}(A_xB_y - A_yB_x)$$

☞ The magnitude of area of the parallelogram formed by the adjacent sides of vectors $\vec{A}$ and $\vec{B}$ equal to $|\vec{A} \times \vec{B}|$

Solved Example

Example 1. $\vec{A}$ is Eastwards and $\vec{B}$ is downwards. Find the direction of $\vec{A} \times \vec{B}$?

Solution : Applying right hand thumb rule we find that $\vec{A} \times \vec{B}$ is along North.

Example 2. If $\vec{A} \cdot \vec{B} = |\vec{A} \times \vec{B}|$, find angle between $\vec{A}$ and $\vec{B}$

Solution : $\vec{A} \cdot \vec{B} = |\vec{A} \times \vec{B}|$ $AB \cos \theta = AB \sin \theta$ $\tan \theta = 1$ $\Rightarrow \theta = 45^0$

Example 3. Two vectors $\vec{A}$ and $\vec{B}$ are inclined to each other at an angle θ. Find a unit vector which is perpendicular to both $\vec{A}$ and $\vec{B}$

Solution : $\vec{A} \times \vec{B} = AB \sin \theta \hat{n}$ $\Rightarrow$ $\hat{n} = \dfrac{\vec{A} \times \vec{B}}{AB \sin \theta}$ here $\hat{n}$ is perpendicular to both $\vec{A}$ and $\vec{B}$.

Example 4. Find $\vec{A} \times \vec{B}$ if $\vec{A} = \hat{i} - 2\hat{j} + 4\hat{k}$ and $\vec{B} = 2\hat{i} - \hat{j} + 2\hat{k}$.

Solution : $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 4 \\ 2 & -1 & 2 \end{vmatrix} = \hat{i}(-4-(-4)) - \hat{j}(2-12) + \hat{k}(-1-(-6)) = 10\hat{j} + 5\hat{k}$

MISCELLANEOUS PROBLEMSS

Problem 1 Find the value of

(a) $\sin(-\theta)$ (b) $\cos(-\theta)$ (c) $\tan(-\theta)$ (d) $\cos(\dfrac{\pi}{2} - \theta)$

(e) $\sin(\dfrac{\pi}{2} + \theta)$ (f) $\cos(\dfrac{\pi}{2} + \theta)$ (g) $\sin(\pi - \theta)$ (h) $\cos(\pi - \theta)$

(i) $\sin(\dfrac{3\pi}{2} - \theta)$ (j) $\cos(\dfrac{3\pi}{2} - \theta)$ (k) $\sin(\dfrac{3\pi}{2} + \theta)$ (l) $\cos(\dfrac{3\pi}{2} + \theta)$

(m) $\tan(\dfrac{\pi}{2} - \theta)$ (n) $\cot(\dfrac{\pi}{2} - \theta)$

Answers :
(a) $-\sin\theta$	(b) $\cos\theta$	(c) $-\tan\theta$	(d) $\sin\theta$
(e) $\cos\theta$	(f) $-\sin\theta$	(g) $\sin\theta$	(h) $-\cos\theta$
(i) $-\cos\theta$	(j) $-\sin\theta$	(k) $-\cos\theta$	(l) $\sin\theta$
(m) $\cot\theta$	(n) $\tan\theta$		

Problem 2 (i) For what value of m the vector $\vec{A} = 2\hat{i} + 3\hat{j} - 6\hat{k}$ is perpendicular to $\vec{B} = 3\hat{i} - m\hat{j} + 6\hat{k}$

(ii) Find the components of vector $\vec{A} = 2\hat{i} + 3\hat{j}$ along the direction of $\hat{i} + \hat{j}$?

Answers : (i) m = –10   (ii) $\dfrac{5}{\sqrt{2}}$.

Problem 3 (i) $\vec{A}$ is North–East and $\vec{B}$ is down wards, find the direction of $\vec{A} \times \vec{B}$.

(ii) Find $\vec{B} \times \vec{A}$ if $\vec{A} = 3\hat{i} - 2\hat{j} + 6\hat{k}$ and $\vec{B} = \hat{i} - \hat{j} + \hat{k}$.

Answers : (i) North - West.   (ii) $-4\hat{i} - 3\hat{j} + \hat{k}$

Exercise-1

✍ **Marked Questions can be used as Revision Questions.**
* Marked Questions may have more than one correct option.

PART - I : FUNCTION & DIFFERENTIATION

Section (A) : Trigonometry and Function

A-1. $f(x) = \cos x + \sin x$. Find $f(\pi/2)$

A-2. If $f(x) = 4x + 3$. Find $f(f(2))$

A-3. $\tan 15°$ is equilvalent to :

(A) $(2 - \sqrt{3})$ (B) $(5 + \sqrt{3})$ (C) $\left(\dfrac{5 - \sqrt{3}}{2}\right)$ (D) $\left(\dfrac{5 + \sqrt{3}}{2}\right)$

A-4. $\sin^2\theta$ is equivalent to :

(A) $\left(\dfrac{1 + \cos\theta}{2}\right)$ (B) $\left(\dfrac{1 + \cos 2\theta}{2}\right)$ (C) $\left(\dfrac{1 - \cos 2\theta}{2}\right)$ (D) $\left(\dfrac{\cos 2\theta - 1}{2}\right)$

A-5. $\sin A \cdot \sin(A + B)$ is equal to

(A) $\cos^2 A \cdot \cos B + \sin A \sin^2 B$ (B) $\sin^2 A \cdot \dfrac{1}{2} \cos B + \cos 2A \cdot \sin B$

(C) $\sin^2 A \cdot \cos B + \dfrac{1}{2} \sin 2A \cdot \sin B$ (D) $\sin^2 A \cdot \sin B + \cos A \cos^2 B$

A-6*.✍ $-\sin\theta$ is equivelent to :

(A) $\cos\left(\dfrac{\pi}{2} + \theta\right)$ (B) $\cos\left(\dfrac{\pi}{2} - \theta\right)$ (C) $\sin(\theta - \pi)$ (D) $\sin(\pi + \theta)$

A-7*.✍ If $x_1 = 8\sin\theta$ and $x_2 = 6\cos\theta$ then
(A) $(x_1 + x_2)_{max} = 10$ (B) $x_1 + x_2 = 10\sin(\theta + 37°)$
(C) $x_1 x_2 = 24 \sin 2\theta$ (D) $\dfrac{x_1}{x_2} = \dfrac{4}{3}\tan\theta$

Section (B) : Differentiation of Elementry Functions
Find the derivative of given functions w.r.t. corresponding independent variable.
B-1. $y = x^2 + x + 8$ **B-2.** $y = \tan x + \cot x$

Find the first derivative & second derivative of given functions w.r.t. corresponding independent variable.
B-3. $y = \sin x + \cos x$ **B-4.** $y = \square nx + e^x$

Section (C) : Differentiation by Product rule
Find derivative of given functions w.r.t. the independent variable x.
C-1. $y = e^x \square nx$ **C-2.** $y = \sin x \cos x$

Section (D) : Differentiation by Quotient rule
Find derivative of given functions w.r.t. the independent variable.

D-1. $y = \dfrac{2x + 5}{3x - 2}$ **D-2.** $y = \dfrac{\ell nx}{x}$ **D-3.**✍ $y = (\sec x + \tan x)(\sec x - \tan x)$

D-4. Suppose u and v are functions of x that are differentiable at x = 0 and that
$u(0) = 5$, $u'(0) = -3$, $v(0) = -1$ $v'(0) = 2$
Find the values of the following derivatives at x = 0.

(a) $\dfrac{d}{dx}(uv)$ (b) $\dfrac{d}{dx}\left(\dfrac{u}{v}\right)$ (c) $\dfrac{d}{dx}\left(\dfrac{v}{u}\right)$ (d) $\dfrac{d}{dx}(7v - 2u)$

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Section (E) : Differentiation by Chain rule

Find $\dfrac{dy}{dx}$ as a function of x

E-1. $y = \sin 5x$ **E-2.** $y = 2\sin(\omega x + \phi)$ where ω and ϕ constants

E-3. $y = (4 - 3x)^9$

Section (F) : Differentiation of Implicit functions

Find $\dfrac{dy}{dx}$

F-1. $(x + y)^2 = 4$ **F-2.** $x^2y + xy^2 = 6$

Section (G) : Differentiation as a rate measurement

G-1. Suppose that the radius r and area $A = \pi r^2$ of a circle are differentiable functions of t. Write an equation that relates dA/dt to dr/dt.

G-2. Suppose that the radius r and surface area $S = 4\pi r^2$ of a sphere are differentiable functions of t. Write an equation that relates $\dfrac{ds}{dt}$ to $\dfrac{dr}{dt}$.

Section (H) : Maxima & Minima

H-1. Particle's position as a function of time is given by $x = -t^2 + 4t + 4$ find the maximum value of position co-ordinate of particle.

H-2. Find the values of function $2x^3 - 15x^2 + 36x + 11$ at the points of maximum and minimum

Section (I) : Miscellaneous

Given $y = f(u)$ and $u = g(x)$, find $\dfrac{dy}{dx}$

I-1. $y = 2u^3$, $u = 8x - 1$ **I-2.** $y = \sin u$, $u = 3x + 1$

I-3. $y = 6u - 9$, $u = (1/2)x^4$ **I-4.** $y = \cos u$, $u = -x/3$

PART - II : INTEGRATION

Section (A) : Integration of elementry functions

Find integrals of given functions

A-1. $x^2 - 2x + 1$ **A-2.** $\sqrt[3]{x} + \dfrac{1}{\sqrt[3]{x}}$ **A-3.** $\sec^2 x$ **A-4.** $\csc^2 x$

A-5. $\sec x \tan x$ **A-6.** $\dfrac{1}{3x}$

Section (B) : Integration by substitution method

Integrate by using the substitution suggested in bracket.

B-1. $\int x \sin(2x^2)\, dx$, (use, $u = 2x^2$)   **B-2.** $\int \sec 2t \tan 2t\, dt$, (use, $u = 2t$)

Integrate by using a suitable substitution

B-3. $\int \dfrac{3}{(2-x)^2}\, dx$ **B-4.** $\int \sin(8z - 5)\, dz$

Section (C) : Definite integration

C-1. $\displaystyle\int_{-4}^{-1} \dfrac{\pi}{2}\, d\theta$ **C-2.** $\displaystyle\int_{\sqrt{2}}^{5\sqrt{2}} r\, dr$ **C-3.** $\displaystyle\int_{0}^{1} e^x\, dx$

Section (D) : Calculation of area

Use a definite integral to find the area of the region between the given curve and the x–axis on the interval [0, b]

D-1. $y = 2x$ **D-2.** $y = \dfrac{x}{2} + 1$

Use a definite integral to find the area of the region between the given curve and the x–axis on the interval $[0, \pi]$

D-3. $y = \sin x$
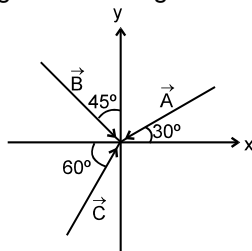
D-4. $y = \sin^2 x$

Objective Questions

D-5*. $I = \int\limits_0^\pi \sin(\theta + \phi)d\theta$, where ϕ is non zero constant then the value of I :

(A) may be positive   (B) may be negative   (C) may be zero   (D) always zero if $\phi = \pi/4$
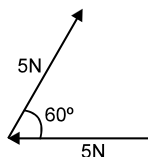
PART - III : VECTOR

Section (A) : Definition of vector & angle between vectors

A-1. Vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ are shown in figure. Find angle between

(i) $\vec{A}$ and $\vec{B}$,   (ii) $\vec{A}$ and $\vec{C}$,   (iii) $\vec{B}$ and $\vec{C}$.

A-2. The forces, each numerically equal to 5 N, are acting as shown in the Figure. Find the angle between forces?

A-3. Rain is falling vertically downwards with a speed 5 m/s. If unit vector along upward is defined as $\hat{j}$, represent velocity of rain in vector form.

Section (B) : Addition of Vectors

B-1. A man walks 40 m North, then 30 m East and then 40 m South. Find the displacement from the starting point?

B-2. A vector of magnitude 30 and direction eastwards is added with another vector of magnitude 40 and direction Northwards. Find the magnitude and direction of resultant with the east.

B-3. Two vectors $\vec{a}$ and $\vec{b}$ inclined at an angle θ w.r.t. each other have a resultant $\vec{c}$ which makes an angle β with $\vec{a}$. If the directions of $\vec{a}$ and $\vec{b}$ are interchanged, then the resultant will have the same
(A) magnitude   (B) direction
(C) magnitude as well as direction   (D) neither magnitude nor direction

B-4. Two vectors $\vec{A}$ and $\vec{B}$ lie in a plane. Another vector $\vec{C}$ lies outside this plane (this plane is not parallel to the plane containing $\vec{A}$ and $\vec{B}$). The resultant $\vec{A} + \vec{B} + \vec{C}$ of these three vectors
(A) can be zero   (B) cannot be zero
(C) lies in the plane of $\vec{A}$ & $\vec{B}$ (D) lies in the plane of $\vec{A}$ & $\vec{A} + \vec{B}$

B-5. The vector sum of the forces of 10 N and 6 N can be
(A) 2 N   (B) 8 N   (C) 18 N   (D) 20 N.

B-6. A set of vectors taken in a given order gives a closed polygon. Then the resultant of these vectors is a
(A) scalar quantity   (B) pseudo vector   (C) unit vector   (D) null vector.

B-7. The vector sum of two force P and Q is minimum when the angle θ between their positive directions, is
(A) $\pi/4$ (B) $\pi/3$ (C) $\pi/2$ (D) π.

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B-8. The vector sum of two vectors $\vec{A}$ and $\vec{B}$ is maximum, then the angle θ between two vectors is -
(A) 0º   (B) 30°   (C) 45°   (D) 60°

B-9. Given : $\vec{C} = \vec{A} + \vec{B}$. Also, the magnitude of $\vec{A}$, $\vec{B}$ and $\vec{C}$ are 12, 5 and 13 units respectively. The angle between $\vec{A}$ and $\vec{B}$ is
(A) 0º   (B) $\pi/4$ (C) $\pi/2$ (D) π.

B-10. If $\vec{P} + \vec{Q} = \vec{P} - \vec{Q}$ and θ is the angle between $\vec{P}$ and $\vec{Q}$, then
(A) $\theta = 0$º   (B) $\theta = 90$º   (C) P = 0   (D) Q = 0

B-11. The sum and difference of two perpendicular vectors of equal lengths are
(A) of equal lengths and have an acute angle between them
(B) of equal length and have an obtuse angle between them
(C) also perpendicular to each other and are of different lengths
(D) also perpendicular to each other and are of equal lengths.

Section (C) : Resolution of Vectors

C-1. Find the magnitude of $3\hat{i} + 2\hat{j} + \hat{k}$?

C-2. What are the x and the y components of a 25 m displacement at an angle of 210º with the x-axis (anti clockwise)?
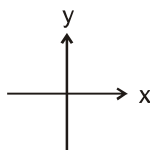
C-3. One of the rectangular components of a velocity of 60 km h^{-1} is 30 km h^{-1}. Find other rectangular component?

C-4. If $0.5\hat{i} + 0.8\hat{j} + C\hat{k}$ is a unit vector. Find the value of C

C-5. The rectangular components of a vector are (2, 2). The corresponding rectangular components of another vector are (1, $\sqrt{3}$). Find the angle between the two vectors

C-6. The x and y components of a force are 2 N and – 3 N. The force is
(A) $2\hat{i} - 3\hat{j}$ (B) $2\hat{i} + 3\hat{j}$ (C) $-2\hat{i} - 3\hat{j}$ (D) $3\hat{i} + 2\hat{j}$

C-7. The vector joining the points A(1, 1, –1) and B(2, –3, 4) and pointing from A to B is -
(A) $-\hat{i} + 4\hat{j} - 5\hat{k}$ (B) $\hat{i} + 4\hat{j} + 5\hat{k}$ (C) $\hat{i} - 4\hat{j} + 5\hat{k}$ (D) $-\hat{i} - 4\hat{j} - 5\hat{k}$.

Section (D) : Products of Vectors

D-1. If $\vec{A} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{B} = 2\hat{i} + \hat{j}$ find (a) $\vec{A} \cdot \vec{B}$ (b) $\vec{A} \times \vec{B}$

D-2. If $|\vec{A}| = 4$, $|\vec{B}| = 3$ and $\theta = 60°$ in the figure, Find (a) $\vec{A} \cdot \vec{B}$ (b) $|\vec{A} \times \vec{B}|$
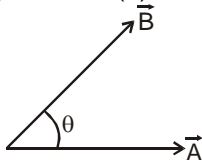
D-3. Three non zero vectors $\vec{A}$, $\vec{B}$ & $\vec{C}$ satisfy the relation $\vec{A} \cdot \vec{B} = 0$ & $\vec{A} \cdot \vec{C} = 0$. Then $\vec{A}$ can be parallel to:
(A) $\vec{B}$ (B) $\vec{C}$ (C) $\vec{B} \cdot \vec{C}$ (D) $\vec{B} \times \vec{C}$

D-4.* The magnitude of scalar product of two vectors is 8 and that of vector product is $8\sqrt{3}$. The angle between them is :
(A) 30º   (B) 60º   (C) 120º   (D) 150º

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Exercise-2

✍ **Marked Questions can be used as Revision Questions.**
* Marked Questions may have more than one correct option.

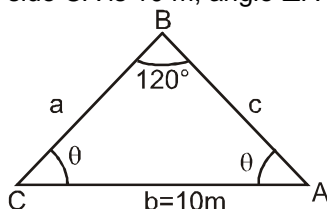# PART - I : FUNCTION & DIFFERENTIATION

1.✍ If $f(x) = \dfrac{x-1}{x+1}$ then find $f\{f(x)\}$

2. $y = f(x) = \dfrac{2x-3}{3x-2}$. Find $f(y)$

Objective Questions

3.✍ For a triangle shown in the figure, side CA is 10 m, angle $\angle A$ and angle $\angle C$ are equal then :

(A) side a = side c = 10m
(B) side $a \ne$ side c

(C) side a = side c = $\dfrac{10\sqrt{3}}{3}$ m
(D) side a = side c = $\dfrac{10}{\sqrt{2}}$ m

4*. If $y_1 = A\sin\theta_1$ and $y_2 = A\sin\theta_2$ then

(A) $y_1 + y_2 = 2A \sin\left(\dfrac{\theta_1 + \theta_2}{2}\right) \cos\left(\dfrac{\theta_1 - \theta_2}{2}\right)$

(B) $y_1 + y_2 = 2A\sin\theta_1 \sin\theta_2$

(C) $y_1 - y_2 = 2A \sin\left(\dfrac{\theta_1 - \theta_2}{2}\right) \cos\left(\dfrac{\theta_1 + \theta_2}{2}\right)$

(D) $y_1 \cdot y_2 = -2A^2 \cos\left(\dfrac{\pi}{2} + \theta_1\right) \cdot \cos\left(\dfrac{\pi}{2} - \theta_2\right)$

5*. Which of following are true
(A) sin 37° + cos 37° = sin 53° + cos53°
(B) sin 37° – cos 37° = cos 53° – sin 53°
(C) tan 37° + 1 = tan 53° – 1
(D) tan 37° × tan 53° = 1

6*.✍ If $R^2 = A^2 + B^2 + 2AB \cos\theta$, if $|A| = |B|$ then value of magnitude of **R** is equivalent to :
(A) $2A\cos\theta$ (B) $A \cos\theta/2$ (C) $2A\cos\theta/2$ (D) $2B\cos\theta/2$

Find the first derivative and second derivative of given functions w.r.t. the independent variable x.

7.✍ $y = \Box nx^2 + \sin x$ **8.** $y = \sqrt[7]{x} + \tan x$

Find derivative of given functions w.r.t. the corresponding independent variable.

9. $y = \left(x + \dfrac{1}{x}\right)\left(x - \dfrac{1}{x} + 1\right)$ **10.**✍ $r = (1 + \sec\theta)\sin\theta$

Find derivative of given functions w.r.t. the respective independent variable.

11. $y = \dfrac{\cot x}{1 + \cot x}$ **12.**✍ $\dfrac{\ell nx + e^x}{\tan x}$

Find $\dfrac{dy}{dx}$ as a function of x

13. $y = \sin^3 x + \sin 3x$ **14.** $\sin^2(x^2 + 1)$

15.✍ $q = \sqrt{2r - r^2}$, find $\dfrac{dq}{dr}$

Find $\dfrac{dy}{dx}$

16. $x^3 + y^3 = 18xy$

17. The radius r and height h of a circular cylinder are related to the cylinder's volume V by the formula $V = \pi r^2 h$.

(a) If height is increasing at a rate of 5 m/s while radius is constant, Find rate of increase of volume of cylinder.

(b) If radius is increasing at a rate of 5 m/s while height is constant, Find rate of increase of volume of cylinder.

(c) If height is increasing at a rate of 5 m/s and radius is increasing at a rate of 5 m/s, Find rate of increase of volume of cylinder.

18. Find two positive numbers x & y such that x + y = 60 and xy is maximum -

19. A sheet of area 40 m² in used to make an open tank with a square base, then find the dimensions of the base such that volume of this tank is maximum.

PART - II : INTEGRATION

Find integrals of given functions.

1. $\int x^{-3}(x+1)\,dx$ **2.** $\int (1-\cot^2 x)\,dx$ **3.** $\int \cos\theta(\tan\theta+\sec\theta)\,d\theta$

Integrate by using the substitution suggested in bracket

4. $\int 12(y^4 + 4y^2 + 1)^2(y^3 + 2y)\,dy$, (use, $u = y^4 + 4y^2 + 1$)

5. $\int \dfrac{dx}{\sqrt{5x+8}}$

(a) Using $u = 5x + 8$ (b) Using $u = \sqrt{5x+8}$

Integrate by using suitable substitution.

6. $\int \sqrt{3-2s}\,ds$ **7.** $\int \sec^2(3x+2)\,dx$ **8.** $\int \csc\left(\dfrac{\upsilon-\pi}{2}\right)\cot\left(\dfrac{\upsilon-\pi}{2}\right)\,d\upsilon$

9. $\int \dfrac{6\cos t}{(2+\sin t)^3}\,dt$ **10.** $\int_{\pi}^{2\pi} \theta\,d\theta$ **11.** $\int_{0}^{\sqrt[3]{7}} x^2\,dx$

12. $\int_{0}^{\sqrt{\pi}} x\sin x^2\,dx$ **13.** $\int_{0}^{1} \dfrac{dx}{3x+2}$

Use a definite integral to find the area of the region between the given curve and the x–axis on the interval [0, b],

14. $y = 3x^2$

PART - III : VECTOR

SUBJECTIVE QUESTIONS

1. Vector $\vec{A}$ points N – E and its magnitude is 3 kg ms⁻¹ it is multiplied by the scalar λ such that $\lambda = -4$ second. Find the direction and magnitude of the new vector quantity. Does it represent the same physical quantity or not ?

2. A force of 30 N is inclined at an angle θ to the horizontal. If its vertical component is 18 N, find the horizontal component & the value of θ.

3. Two vectors acting in the opposite directions have a resultant of 10 units. If they act at right angles to each other, then the resultant is 50 units. Calculate the magnitude of two vectors.

4. The angle θ between directions of forces $\vec{A}$ and $\vec{B}$ is 90º where A = 8 dyne and B = 6 dyne. If the resultant $\vec{R}$ makes an angle α with $\vec{A}$ then find the value of 'α' ?

5. Find the resultant of the three vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ each of magnitude r as shown in figure?
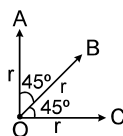
6. If $\vec{A} = 3\hat{i} + 4\hat{j}$ and $\vec{B} = \hat{i} + \hat{j} + 2\hat{k}$ then find out unit vector along $\vec{A} + \vec{B}$.

7. The x and y components of vector $\vec{A}$ are 4m and 6m respectively. The x, y components of vector $\vec{A} + \vec{B}$ are 10m and 9m respectively. Find the length of $\vec{B}$ and angle that $\vec{B}$ makes with the x axis.

OBJECTIVE QUESTIONS

Single choice type

8. A vector is not changed if
(A) it is displaced parallel to itself
(B) it is rotated through an arbitrary angle
(C) it is cross-multiplied by a unit vector
(D) it is multiplied by an arbitrary scalar.

9. If the angle between two forces increases, the magnitude of their resultant
(A) decreases
(B) increases
(C) remains unchanged
(D) first decreases and then increases

10. A car is moving on a straight road due north with a uniform speed of 50 km h⁻¹ when it turns left through 90º. If the speed remains unchanged after turning, the change in the velocity of the car in the turning process is
(A) zero
(B) $50\sqrt{2}$ km h⁻¹ S-W direction
(C) $50\sqrt{2}$ km h⁻¹ N-W direction
(D) 50 km h⁻¹ due west.

11. Which of the following sets of displacements might be capable of bringing a car to its returning point?
(A) 5, 10, 30 and 50 km
(B) 5, 9, 9 and 16 km
(C) 40, 40, 90 and 200 km
(D) 10, 20, 40 and 90 km

12. When two vector $\vec{a}$ and $\vec{b}$ are added, the magnitude of the resultant vector is always
(A) greater than (a + b)
(B) less than or equal to (a + b)
(C) less than (a + b)
(D) equal to (a + b)

13. If $|\vec{A} + \vec{B}| = |\vec{A}| = |\vec{B}|$, then the angle between $\vec{A}$ and $\vec{B}$ is
(A) 0º
(B) 60º
(C) 90º
(D) 120º.

14. Given : $\vec{a} + \vec{b} + \vec{c} = 0$. Out of the three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ two are equal in magnitude. The magnitude of the third vector is $\sqrt{2}$ times that of either of the two having equal magnitude. The angles between the vectors are:
(A) 90º, 135º, 135º
(B) 30º, 60º, 90º
(C) 45º, 45º, 90º
(D) 45º, 60º, 90º

15. Vector $\vec{A}$ is of length 2 cm and is 60º above the x-axis in the first quadrant. Vector $\vec{B}$ is of length 2 cm and 60º below the x-axis in the fourth quadrant. The sum $\vec{A} + \vec{B}$ is a vector of magnitude -
(A) 2 along + y-axis
(B) 2 along + x-axis
(C) 1 along – x axis
(D) 2 along – x axis

16. Six forces, 9.81 N each, acting at a point are coplanar. If the angles between neighboring forces are equal, then the resultant is
(A) 0 N
(B) 9.81 N
(C) 2 × 9.81 N
(D) 3 × 9.81 N.

17. A vector $\vec{A}$ points vertically downward & $\vec{B}$ points towards east, then the vector product $\vec{A} \times \vec{B}$ is
(A) along west
(B) along east
(C) zero
(D) along south

More than one choice type

18. In the Figure which of the ways indicated for combining the x and y components of vector a are proper to determine that vector?

(A) (iii) (B) (iv) (C) (vi) (D) (i), (ii) and (v).
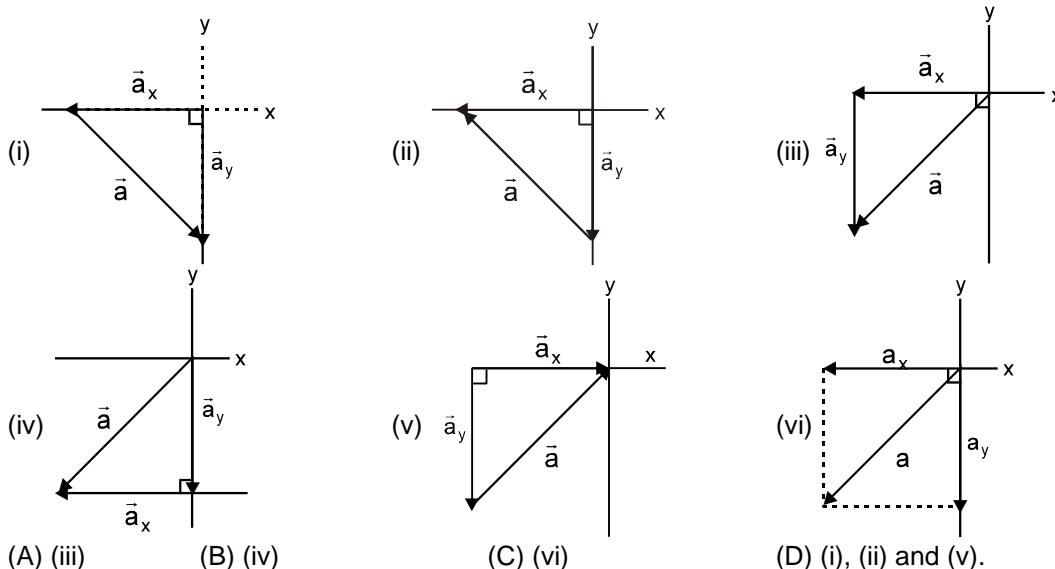
19. Let $\vec{a}$ and $\vec{b}$ be two non-null vectors such that $|\vec{a}+\vec{b}| = |\vec{a}-2\vec{b}|$. Then the value of $\dfrac{|\vec{a}|}{|\vec{b}|}$ may be :

(A) 1/4 (B) 1/8 (C) 1 (D) 2

Exercise-3

✎ **Marked Questions can be used as Revision Questions.**

PART - I : MATCH THE COLUMN

1.✎ Match the integrals (given in column-II) with the given functions (in column-I)

Column-I

(A) $\int \sec x \tan x \, dx$

(B) $\int \operatorname{cosec} Kx \cot Kx \, dx$

(C) $\int \operatorname{cosec}^2 Kx \, dx$

(D) $\int \cos Kx \, dx$

Column-II

(p) $-\dfrac{\operatorname{cosec} Kx}{K} + C$

(q) $-\dfrac{\cot Kx}{K} + C$

(r) $\sec x + C$

(s) $\dfrac{\sin Kx}{K} + C$

2.✎ Match the statements given in column-I with statements given in column-II

Column-I

(A) if $|\vec{A}| = |\vec{B}|$ and $|\vec{A} + \vec{B}| = |\vec{A}|$ then angle between $\vec{A}$ and $\vec{B}$ is

(B) Magnitude of resultant of two forces $|\vec{F}_1| = 8N$ and $|\vec{F}_2| = 4$ N may be

(C) Angle between $\vec{A} = 2\hat{i} + 2\hat{j}$ & $\vec{B} = 3\hat{k}$ is

(D) Magnitude of resultant of vectors $\vec{A} = 2\hat{i} + \hat{j}$ & $\vec{B} = 3\hat{k}$ is

Column-II

(p) 90°

(q) 120°

(r) 12 N

(s) $\sqrt{14}$

PART - II : COMPREHENSION

COMPREHENSION-1

A particle is moving along positive x-axis. Its position varies as $x = t^3 - 3t^2 + 12t + 20$, where x is in meters and t is in seconds.

1. Initial velocity of the particle is.
(A) 1 m/s   (B) 3 m/s   (C) 12 m/s   (D) 20 m/s

2. Initial acceleration of the particle is
(A) Zero   (B) 1 m/s^2 (C) -3 m/s^2 (D) -6 m/s^2

3. Velocity of the particle when its acceleration zero is
(A) 1 m/s   (B) 3 m/s   (C) 6 m/s   (D) 9 m/s

COMPREHENSION-2

Two forces $\vec{F}_1 = 2\hat{i} + 2\hat{j}$ N and $\vec{F}_2 = 3\hat{j} + 4\hat{k}$ N are acting on a particle.

4. The resultant force acting on particle is :
(A) $2\hat{i} + 5\hat{j} + 4\hat{k}$ (B) $2\hat{i} - 5\hat{j} - 4\hat{k}$ (C) $\hat{i} - 3\hat{j} - 2\hat{k}$ (D) $\hat{i} - \hat{j} - \hat{k}$

5. The angle between $\vec{F}_1$ & $\vec{F}_2$ is :
(A) $\theta = \cos^{-1}\left(\dfrac{3}{2\sqrt{5}}\right)$ (B) $\theta = \cos^{-1}\left(\dfrac{3}{5\sqrt{2}}\right)$ (C) $\theta = \cos^{-1}\left(\dfrac{2}{3\sqrt{5}}\right)$ (D) $\theta = \cos^{-1}\left(\dfrac{\sqrt{3}}{5}\right)$

6. The component of force $\vec{F}_1$ along force $\vec{F}_2$ is :
(A) $\dfrac{5}{6}$ (B) $\dfrac{5}{3}$ (C) $\dfrac{6}{5}$ (D) $\dfrac{5}{2}$

PART - III : ASSERTION / REASON

1. **Statement-1 :** A vector is a quantity that has both magnitude and direction and obeys the triangle law of addition.
Statement-2 : The magnitude of the resultant vector of two given vectors can never be less than the magnitude of any of the given vector.
(A) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1.
(B) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1
(C) Statement-1 is True, Statement-2 is False
(D) Statement-1 is False, Statement-2 is True

2. **Statement-1 :** If the rectangular components of a force are 8 N and 6N, then the magnitude of the force is 10N.
Statement-2 : If $|\vec{A}| = |\vec{B}| = 1$ then $|\vec{A} \times \vec{B}|^2 + |\vec{A} \cdot \vec{B}|^2 = 1$.
(A) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1.
(B) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1
(C) Statement-1 is True, Statement-2 is False
(D) Statement-1 is False, Statement-2 is True

3. **Statement-1 :** If three non zero vectors $\vec{A}$, $\vec{B}$ **and** $\vec{C}$ satisfy the relation $\vec{A} \cdot \vec{B} = 0$ & $\vec{A} \cdot \vec{C} = 0$ then the vector $\vec{A}$ is parallel to $\vec{B} \times \vec{C}$.
Statement-2 : $\vec{A} \perp \vec{B}$ and $\vec{A} \perp \vec{C}$ and $\vec{B} \times \vec{C} \neq 0$ hence $\vec{A}$ is perpendicular to plane formed by $\vec{B}$ and $\vec{C}$.
(A) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1.
(B) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1
(C) Statement-1 is True, Statement-2 is False

(D) Statement-1 is False, Statement-2 is True

4. **Statement-1 :** The minimum number of non-zero vectors of unequal magnitude required to produce zero resultant is three.
Statement-2 : Three vectors of unequal magnitude which can be represented by the three sides of a triangle taken in order, produce zero resultant.
(A) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1.
(B) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1
(C) Statement-1 is True, Statement-2 is False
(D) Statement-1 is False, Statement-2 is True

5. **Statement-1 :** The angle between the two vectors $\left(\hat{i} + \hat{j}\right)$ and $\left(\hat{k}\right)$ is $\dfrac{\pi}{2}$ radian.

Statement-2 : Angle between two vectors $\left(\hat{i} + \hat{j}\right)$ and $\left(\hat{k}\right)$ is given by $\theta = \cos^{-1}\left(\dfrac{\vec{A}.\vec{B}}{AB}\right)$.

(A) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1.
(B) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1
(C) Statement-1 is True, Statement-2 is False
(D) Statement-1 is False, Statement-2 is True

6. **Statement-1 :** Distance is a scalar quantity.
Statement-2 : Distance is the length of path transversed.
(A) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1.
(B) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1
(C) Statement-1 is True, Statement-2 is False
(D) Statement-1 is False, Statement-2 is True

PART - IV : TRUE / FALSE

1. **State True or False**
(i) $f(x) = -f'(x)$ for some function f.
(ii) $f(x) = f'(x)$ for some function f.

(iii) If $\vec{A}$ & $\vec{B}$ are two force vectors then $\vec{A}.\vec{B} = \vec{B}.\vec{A}$

(iv) If $\vec{A}$ & $\vec{B}$ are two non-zero force vectors then $\vec{A} \times \vec{B} = \vec{B} \times \vec{A}$

(v) If the vector product of two non-zero vectors vanishes, the vectors are collinear.

PART - V : FILL IN THE BLANKS

Fill in the blanks

1. The magnitude of sum of three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ representing the sides of a cube of length A is equal to ....
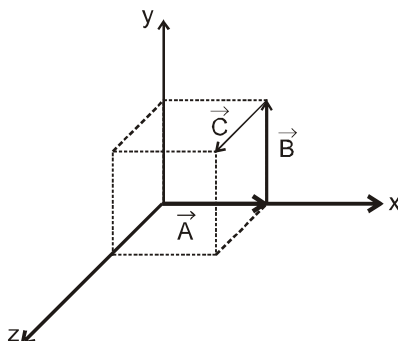
2. If $\vec{A} = 3\hat{i} + 4\hat{j}$ and $\vec{B} = 7\hat{i} + 24\hat{j}$, then the vector having the same magnitude as $\vec{B}$ and parallel to $\vec{A}$ is ....

3. If $\vec{A} \| \vec{B}$ then $\vec{A} \times \vec{B} =$ ....

4. The magnitude of area of the parallelogram formed by the adjacent sides of vectors $\vec{A} = 3\hat{i} + 2\hat{j}$ and $\vec{B} = 2\hat{i} - 4\hat{k}$ is ....

5. If $\vec{A}$ is .... to $\vec{B}$, then $\vec{A} \cdot \vec{B} = 0$

6. The vector $\vec{A} = \hat{i} + \hat{j}$, where $\hat{i}$ and $\hat{j}$ are unit vectors along x-axis and y-axis respectively, makes an angle of .... degree with x-axis.

7. Two vectors $\vec{A}$ and $\vec{B}$ are defined as $\vec{A} = \alpha\hat{i}$ and $\vec{B} = \alpha(\cos\omega t\,\hat{i} + \sin\omega t\,\hat{j})$, where α is a constant and $\omega = \pi/6$ rad s^{-1}. If $|\vec{A} + \vec{B}| = \sqrt{3}\,|\vec{A} - \vec{B}|$ at time t = τ for the first time, the value of τ, in second, is _____.

[JEE (Advanced) 2018; P-1, 3/60]

Answers

EXERCISE-1

PART - I

Section (A) :

A-1. 1 **A-2.** 47 **A-3.** (A)

A-4. (C) **A-5.** (C) **A-6.** (ACD)

A-7. (ABCD)

Section (B) :

B-1. $\dfrac{dy}{dx} = 2x + 1$

B-2. $\sec^2 x - \operatorname{cosec}^2 x$

B-3. $\dfrac{dy}{dx} = \cos x - \sin x$, $\dfrac{d^2y}{dx^2} = -\sin x - \cos x$

B-4. $\dfrac{dy}{dx} = \dfrac{1}{x} + e^x$, $\dfrac{d^2y}{dx^2} = -\dfrac{1}{x^2} + e^x$

Section (C) :

C-1. $e^x\,\square nx + \dfrac{e^x}{x}$ **C-2.** $\cos^2 x - \sin^2 x$

Section (D) :

D-1. $y' = \dfrac{-19}{(3x-2)^2}$ **D-2.** $\dfrac{1}{x^2} - \dfrac{\ell nx}{x^2}$

D-3. $\dfrac{dy}{dx} = 0$

D-4. (a) 13 (b) -7 (c) $\dfrac{7}{25}$ (d) 20

Section (E) :

E-1. 5 cos 5x

E-2. $2\omega\cos(\omega x + \phi)$

E-3. $\dfrac{dy}{dx} = -27(4 - 3x)^8$

Section (F) :

F-1. dy/dx = −1 **F-2.** $\dfrac{-2xy - y^2}{x^2 + 2xy}$

Section (G) :

G-1. $\dfrac{dA}{dt} = 2\pi r\dfrac{dr}{dt}$. **G-2.** $\dfrac{ds}{dt} = 8\pi r\dfrac{dr}{dt}$

Section (H) :

H-1. 8 **H-2.** $y_{max} = 39$, $y_{min} = 38$

Section (I) :

I-1. $48(8x - 1)^2$ **I-2.** 3 cos (3x + 1)

I-3. $12x^3$. **I-4.** $\dfrac{dy}{dx} = -\dfrac{1}{3}\sin\dfrac{x}{3}$

PART - II

Section (A) :

A-1. $\dfrac{x^3}{3} - x^2 + x + c$ **A-2.** $\dfrac{3x^{4/3}}{4} + \dfrac{3x^{2/3}}{2} + c$

A-3. tan x + c **A-4.** − cot x + c

A-5. sec x + c **A-6.** $\dfrac{1}{3}\ell nx + c$

Section (B) :

B-1. $-\dfrac{1}{4}\cos(2x^2) + C$ **B-2.** $\dfrac{1}{2}\sec 2t + C$

B-3. $\dfrac{3}{2-x} + C$ **B-4.** $-\dfrac{\cos(8z-5)}{8} + C$

Section (C) :

C-1. $\dfrac{3\pi}{2}$ **C-2.** 24 **C-3.** e − 1

Section (D) :

D-1. Using n subintervals of length $\Delta x = \dfrac{b}{n}$ and right– endpoint values :

Area = $\int\limits_0^b 2x \, dx = b^2$ units

D-2. $\dfrac{b^2}{4} + b = \dfrac{b(4+b)}{4}$ units

D-3. 2 units　　　**D-4.** $\pi/2$ units

D-5. (ABC)

PART - III

Section (A) :

A-1. (i) 105º, (ii) 150º, (iii) 105º.

A-2. 120º　　　**A-3.** $\vec{V}_R = -5\hat{j}$

Section (B) :

B-1. 30 m East

B-2. 50, 53º with East

B-3. (A)　　**B-4.** (B)　　**B-5.** (B)

B-6. (D)　　**B-7.** (D)　　**B-8.** (A)

B-9. (C)　　**B-10.** (D)　　**B-11.** (D)

Section (C) :

C-1. $\sqrt{14}$

C-2. $-25\cos 30º$ and $-25\sin 30º$

C-3. $30\sqrt{3}$ km h^{-1}.　　**C-4.** $\pm\dfrac{\sqrt{11}}{10}$

C-5. 15º　**C-6.** (A)　**C-7.** (C)

Section (D) :

D-1. (a) 3　　　(b) $-\hat{i} + 2\hat{j} - \hat{k}$

D-2. (a) 6　　　(b) $6\sqrt{3}$

D-3. (D)　**D-4.** (B)

EXERCISE-2
PART - I

1. $-1/x$　**2.** X　**3.** (C)

4. (AC)　**5.** (ABD)　**6.** (CD)

7. $\dfrac{dy}{dx} = \dfrac{2}{x} + \cos x,\ \dfrac{d^2y}{dx^2} = \dfrac{-2}{x^2} - \sin x$

8. $\dfrac{dy}{dx} = \dfrac{x^{-\frac{6}{7}}}{7} + \sec^2 x,\ \dfrac{d^2y}{dx^2} = \dfrac{-6}{49}x^{-\frac{13}{7}} + 2\tan x\sec^2 x$

9. $\dfrac{dy}{dx} = 1 + 2x + \dfrac{2}{x^3} - \dfrac{1}{x^2}$

10. $dr/d\theta = \cos\theta + \sec^2\theta$

11. $\dfrac{-\csc^2 x}{(1+\cot x)^2}$

12. $\dfrac{\tan x\left(e^x + \dfrac{1}{x}\right) - \sec^2 x(e^x + \ell nx)}{\tan^2 x}$

13. $3\sin^2 x \cos x + 3\cos 3x$

14. $4x \sin(x^2 + 1)\cos(x^2 + 1)$

15. $\dfrac{1-r}{\sqrt{2r - r^2}}$　　**16.** $\dfrac{dy}{dx} = \dfrac{18y - 3x^2}{3y^2 - 18x}$

17. (a) $\dfrac{dV}{dt} = \pi r^2 \dfrac{dh}{dt} = 5\pi r^2$

(b) $\dfrac{dV}{dt} = 2\pi hr\,\dfrac{dr}{dt} = 10\pi rh$

(c) $\dfrac{dV}{dt} = \pi r^2 \dfrac{dh}{dt} + 2\pi hr\dfrac{dr}{dt} = 5\pi r^2 + 10\pi rh$

18. x = 30 & y = 30

19. $x = \sqrt{\dfrac{40}{3}}$ m

PART - II

1. $-\dfrac{1}{x} - \dfrac{1}{2x^2} + C$

2. $2x + \cot x + C$

3. $-\cos\theta + \theta + C$

4. $(y^4 + 4y^2 + 1)^3 + C$

5. $\left[\dfrac{2}{5}\sqrt{5x+8}\right] + C$

6. $-\dfrac{1}{3}(3 - 2s)^{3/2} + C$

7. $\dfrac{1}{3}\tan(3x + 2) + C$

8. $-2\csc\left(\dfrac{\upsilon - \pi}{2}\right) + C$

9. $\dfrac{-3}{(2+\sin t)^2} + C$

10. $\dfrac{3\pi^2}{2}$　**11.** 7/3　**12.** 1

13. $\dfrac{1}{3}\ell n\dfrac{5}{2} = \ell n\left(\dfrac{5}{2}\right)^{\frac{1}{3}}$

14. Using n subintervals of length $\Delta x = b/n$ and right–end point values:

$$\text{Area} = \int_0^b 3x^2 \, dx = b^3$$

PART - III

1. No it does not represent the same physical quantity.

2. 24 N; 37° approx.

3. P = 40; Q = 30   **4.** 37⁰

5. $r(1 + \sqrt{2})$ **6.** $\dfrac{4\hat{i} + 5\hat{j} + 2\hat{k}}{\sqrt{45}}$

7. $3\sqrt{5}$, $\tan^{-1}\dfrac{1}{2}$ **8.** (A)

9. (A)   **10.** (B)   **11.** (B)

12. (B)   **13.** (D)   **14.** (A)

15. (B)   **16.** (A)   **17.** (D)

18. (ABC)   **19.** (D)

EXERCISE-3
PART - I

1. (A) → r, (B) → p, (C) → q, (D) → s

2. (A) → q, (B) → r, (C) → p, (D) → s

PART - II
1. (C)   **2.** (D)   **3.** (D)
4. (A)   **5.** (B)   **6.** (C)

PART - III
1. (C)   **2.** (B)   **3.** (D)
4. (A)   **5.** (A)   **6.** (A)

PART - IV
1. (i) T   (ii) T   (iii) T
(iv) F   (v) T

PART - V
1. $(\sqrt{3})$A   **2.** $15\hat{i} + 20\hat{j}$.

3. Null vector   **4.** $\sqrt{224}$ units

5. Perpendicular.   **6.** 45⁰

7. 2.00

High Level Problems (HLP)

✎ **Marked Questions can be used as Revision Questions.**

PART - I : DIFFERENTIATION

Find the derivative of functions using quotient rule.

1. $g(x) = \dfrac{x^2 - 4}{x + 0.5}$

2. Suppose u and v are differentiable functions of x and that
u (1) = 2, u'(1) = 0 v (1) = 5 v'(1) = – 1.
Find the values of the following derivatives at x = 1.

(a) $\dfrac{d}{dx}$ (uv) (b) $\dfrac{d}{dx}\left(\dfrac{u}{v}\right)$ (c) $\dfrac{d}{dx}\left(\dfrac{v}{u}\right)$ (d) $\dfrac{d}{dx}$ (7v – 2u).

Find $\dfrac{ds}{dt}$.

3. $s = \dfrac{1 + \cosec t}{1 - \cosec t}$ **4.** $s = \dfrac{\sin t}{1 - \cos t}$

Find $\dfrac{dy}{dx}$ as a function of x.

5. $y = \sin^3 x$ **6.** $y = 5 \cos^{-4} x$.

Find the derivatives of the functions

7. $r = (\cosec\theta + \cot\theta)^{-1}$ **8.** $r = - (\sec\theta + \tan\theta)^{-1}$

PART - II : INTEGRATION

Find an antiderivative for each function. Do as many as you can mentally. Check your answer by differentiation.

1. (a) csc x cot x (b) – csc 5x cot 5x (c) $- \pi \csc \dfrac{\pi x}{2} \cot \dfrac{\pi x}{2}$

2. $(1 + 2 \cos x)^2$

Evaluating Integrals.
Check your answers by differentiation.

3. $\displaystyle\int\left(\dfrac{\sqrt{x}}{2} + \dfrac{2}{\sqrt{x}}\right) dx$ **4.** $\displaystyle\int\left(8y - \dfrac{2}{y^{1/4}}\right) dy$ **5.** $\displaystyle\int 2x(1 - x^{-3})\, dx$

6. $\displaystyle\int (-3\csc^2 x)\, dx$ **7.** $\displaystyle\int\left(-\dfrac{\sec^2 x}{3}\right) dx$ **8.** $\displaystyle\int \dfrac{\csc\theta \cot\theta}{2}\, d\theta$

9. $\displaystyle\int \dfrac{2}{5}\sec\theta \tan\theta\, d\theta$ **10.** $\displaystyle\int (4\sec x\tan x - 2\sec^2 x)\, dx$

11. $\displaystyle\int \dfrac{1}{2}(\csc^2 x - \csc x\cot x)\, dx$ **12.** $\displaystyle\int (\sin 2x - \csc^2 x)\, dx$

13. $\displaystyle\int (2\cos 2x - 3\sin 3x)\, dx$ **14.** $\displaystyle\int 4\sin^2 y\, dy$ **15.** $\displaystyle\int \dfrac{\csc\theta}{\csc\theta - \sin\theta}\, d\theta$

Evaluate Integrals by substitution method.

16. $\int \dfrac{1}{\sqrt{5s+4}}\ ds$ **17.** $\int 3y\sqrt{7-3y^2}\ dy$ **18.** $\int \sin^5 \dfrac{x}{3}\cos\dfrac{x}{3}\ dx.$

19. $\int \tan^7 \dfrac{x}{2}\sec^2\dfrac{x}{2}\ dx$ **20.** $\int r^2\left(\dfrac{r^3}{18}-1\right)^5 dr$ **21.** $\int r^4\left(7-\dfrac{r^5}{10}\right)^3 dr$

22. $\int x^{1/3}\sin(x^{4/3}-8)\ dx.$ **23.** $\int \csc\left(\dfrac{\upsilon-\pi}{2}\right)\cot\left(\dfrac{\upsilon-\pi}{2}\right)d\upsilon$

24. $\int \sqrt{\cot y}\ \csc^2 y\ dy$ **25.** $\int \dfrac{\sec z\tan z}{\sqrt{\sec z}}\ dz$ **26.** $\int \dfrac{1}{t^2}\cos\left(\dfrac{1}{t}-1\right)dt$

Find the definite integrals of following Functions

27. $\displaystyle\int_{1/2}^{3/2}(-2x+4)\ dx$ **28.** $\displaystyle\int_{-2}^{1}|x|\ dx$

Evaluate definite integrals of following functions

29. $\displaystyle\int_{0}^{\pi/2}\theta^2\ d\theta$ **30.** $\displaystyle\int_{0}^{3b}x^2\ dx$

PART - III : VECTOR

1. Four forces of magnitudes P, 2P, 3P and 4P act along the four sides of a square ABCD in cyclic order. Use the vector method to find the resultant force.

2. A sail boat sails 2km due East, 5km 37^0 South of East and finally has an unknown displacement. If the final displacement of the boat from the starting point is 6km due East, the third displacement is _____.

3. The resultant of two vectors **u** and **v** is perpendicular to the vector u and its magnitude is equal to half of the magnitude of vector **v**. Find the angle between **u** and **v**.

4. Let the resultant of three forces of magnitude 5N, 12N & 13N acting on a body be zero. If sin 23° = (5/13), find the angle between the 5N force & 13N force.

5. Two vectors $\vec{A}$ & $\vec{B}$ have the same magnitude. Under what circumstances does the vector $\vec{A}+\vec{B}$ have the same magnitude as $|\vec{A}|$ or $|\vec{B}|$. When does the vector difference $\vec{A}-\vec{B}$ have this magnitude?

6. The resultant of $\vec{P}$ and $\vec{Q}$ is $\vec{R}$. If magnitude of $\vec{Q}$ is doubled, magnitude of resultant is also doubled, when direction of $\vec{Q}$ is reversed from initial condition then magnitude of resultant is again doubled, find P : Q : R.

7. If five consecutive sides of a regular hexagon represent five unit vectors acting in the same sense, find their resultant vector. (Taking first side on x-axis)

8. Four ants 1, 2, 3 and 4 are pulling a grain with force of magnitudes 3 N, 1N, 2N and $|\vec{F}|$ N as shown in the Fig. Find force $\vec{F}$ if the grain remains in equilibrium under the action of the above forces.
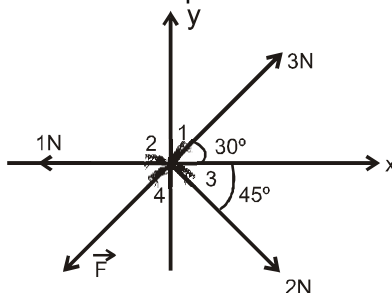
9. An insect moves in a circular path of radius R. If it rotates through an angle θ, find its displacement means $\vec{s} = \Delta\vec{r}$, where $\Delta\vec{r}$ is "the change in position vector".
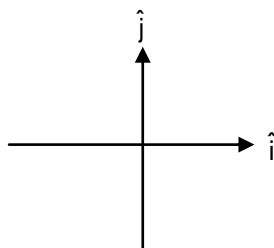
10. Find the vector equation of a line which is parallel to a given vector $\vec{A}$ and passes through a given point P having position vector $\vec{r}_0$.

11. By using the concept of scalar product prove that $|\vec{A}+\vec{B}| = \sqrt{|\vec{A}|^2 + |\vec{B}|^2 + 2|\vec{A}||\vec{B}|\cos\theta}$

12. The vector $\vec{A}$ varies with time as $\vec{A} = t\hat{i} - \sin\pi t\hat{j} + t^2\hat{k}$. Find the derivative of the vector at t = 1.

13. A body hanging from a spring (fig.) is stretched 5 units beyond its rest position and released at time t = 0 to oscillate up and down. Its position at any later time t is s = 5 cos t. What are its velocity and acceleration at time t ?
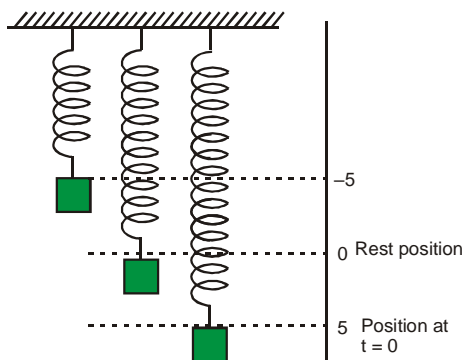
14. A cylinder of radius R is moving down with speed v into water placed in a cylinder of radius 4R as shown. Find the rate by which
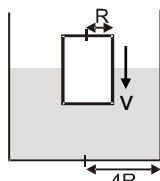
(a) Water surface is rising   (b) Wet surface area of cylinder of radius R is increasing.

15. Find the maximum area of the rectangle that can be inscribed in a circle of radius r ?

HLP Answers

PART - I

1. g' (x) = $\dfrac{x^2 + x + 4}{(x+0.5)^2}$ **2.** (a) –2   (b) $\dfrac{2}{25}$ (c) $-\dfrac{1}{2}$ (d) –7

3. $\dfrac{-2 \cosec t \cot t}{(1-\cosec t)^2}$ **4.** $\dfrac{ds}{dt} = \dfrac{1}{\cos t - 1}$ **5.** 3 sin² x (cos x)

6. $\dfrac{dy}{dx} = 20\sin x \cos^{-5} x$ **7.** $\dfrac{\cosec\theta}{\cot\theta + \cosec\theta}$ **8.** $\dfrac{\sec\theta}{\sec\theta + \tan\theta}$

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PART - II

1. (a) $- \csc x + C$ (b) $\dfrac{1}{5} \csc(5x) + C$ (c) $2 \csc\left(\dfrac{\pi x}{2}\right) + C$ **2.** $3x + \sin 2x + 4 \sin x + C$

3. $\dfrac{x^{\frac{3}{2}}}{3} + 4x^{1/2} + C$ **4.** $4y^2 - \dfrac{8}{3} y^{3/4} + C$ **5.** $x^2 + \dfrac{2}{x} + C$

6. $3 \cot x + C$ **7.** $\dfrac{-\tan x}{3} + C$ **8.** $-\dfrac{1}{2} \csc \theta + C$

9. $\dfrac{2}{5} \sec \theta + C$ **10.** $4 \sec x - 2 \tan x + C$ **11.** $-\dfrac{1}{2} \cot x + \dfrac{1}{2} \csc x + C$

12. $-\dfrac{1}{2} \cos 2x + \cot x + C$ **13.** $\sin 2x + \cos 3x + C$ **14.** $2y - \sin 2y + C$

15. $\tan \theta + C$ **16.** $\dfrac{2}{5} (5s + 4)^{1/2} + C$ **17.** $-\dfrac{1}{3} (7 - 3y^2)^{3/2} + C$

18. $\dfrac{1}{2} \sin^6\left(\dfrac{x}{3}\right) + C$ **19.** $\dfrac{1}{4} \tan^8 \dfrac{x}{2} + C$ **20.** $\left(\dfrac{r^3}{18} - 1\right)^6 + C$

21. $-\dfrac{1}{2} \left(7 - \dfrac{r^5}{10}\right)^4 + C$ **22.** $-\dfrac{3}{4} \cos(x^{4/3} - 8) + C$ **23.** $-2 \csc\left(\dfrac{\upsilon - \pi}{2}\right) + C$

24. $-\dfrac{2}{3} (\cot^3 y)^{1/2} + C$ **25.** $2\sqrt{\sec z} + C$ **26.** $-\sin\left(\dfrac{1}{t} - 1\right) + C$

27. Area = 2 square units   **28.** Area = 2.5 square units   **29.** $\dfrac{\pi^3}{24}$ **30.** $9b^3$

PART - III

1. $2\sqrt{2}\, P$ **2.** 3 km in North   **3.** $150°$ **4.** $113°$

5. When the angle between **A** & **B** is 120°; when it is 60°   **6.** $P : Q : R = \sqrt{2} : \sqrt{3} : \sqrt{2}$

7. $\vec{A}_3 = \dfrac{1}{2}(-\hat{i} + \sqrt{3}\hat{j})$

8. $F_4 = \sqrt{x^2 + y^2} = \sqrt{\left(\dfrac{3\sqrt{3}}{2} + \sqrt{2} - 1\right)^2 + \left(\dfrac{3}{2} - \sqrt{2}\right)^2}\ N,\ \phi = \tan^{-1}\left[\dfrac{3 - 2\sqrt{2}}{3\sqrt{3} + 2\sqrt{2} - 2}\right]$

9. The magnitude of displacement $= |\Delta \vec{r}| = 2R \sin \dfrac{\theta}{2}$. The direction of $\Delta \vec{r}$ is given as $\beta = \dfrac{\pi}{2} + \dfrac{\theta}{2}$

10. $\vec{r} = \vec{r}_0 + n\hat{a}$

11. $\vec{A} + \vec{B} = \vec{R}$, say,

Since $|\vec{R}|^2 = \vec{R}.\vec{R} = (\vec{A} + \vec{B}).(\vec{A} + \vec{B})$

Expanding the product, we have $|\vec{R}|^2 = \vec{A}.\vec{A} + \vec{A}.\vec{B} + \vec{B}.\vec{A} + \vec{B}.\vec{B}$

Substituting $\vec{A}.\vec{A} = A^2, \vec{A}.\vec{B} = \vec{B}.\vec{A}$ and $\vec{B}.\vec{B} = |\vec{B}|^2$, we have $|\vec{R}|^2 = |\vec{A}|^2 + |\vec{B}|^2 + 2\vec{A}.\vec{B}$

Hence $|\vec{R}| = |\vec{A} + \vec{B}| = \sqrt{|\vec{A}|^2 + |\vec{B}|^2 + 2|\vec{A}||\vec{B}| \cos\theta}$

12. $\hat{i} + \pi\hat{j} + 2\hat{k}$ **13.** Velocity: $-5 \sin t$, Acceleration: $-5 \cos t$

14. (a) $\dfrac{dh}{dt} = \dfrac{v}{15}$ (b) $\dfrac{32\pi R v}{15}$ **15.** $A_{max} = 2r^2$

SOLUTIONS OF MATHEMATICAL TOOLS
EXERCISE-1
PART - I

Section (A) :

A-1. $f(\pi/2) = \cos \pi/2 + \sin \pi/2 = 1$

A-2. $f(2) = 4 \times 2 + 3 = 11$

$f\{f(2)\} = f(11) = 4 \times 11 + 3 = 47$

A-3. $\tan 15 = \tan(45-30) = \dfrac{\tan 45 - \tan 30}{1 + \tan 45 \tan 30} = \dfrac{1 - \dfrac{1}{\sqrt{3}}}{1 + \dfrac{1}{\sqrt{3}}} = \dfrac{\sqrt{3} - 1}{\sqrt{3} + 1}$

$= \dfrac{(\sqrt{3} - 1)^2}{2} = \dfrac{3 + 1 - 2\sqrt{3}}{2} = \dfrac{4 - 2\sqrt{3}}{2} = 2 - \sqrt{3}$

A-4. $\cos^2 \theta = 1 - 2\sin^2 \theta$

$2\sin^2 \theta = 1 - \cos^2 \theta \Rightarrow \sin^2 \theta = \left(\dfrac{1 - \cos 2\theta}{2} \right)$

A-5. $\sin A \cdot [\sin A \cos B + \cos A \cdot \sin B]$

$\sin^2 A \cdot \cos B + \sin A \cdot \cos A \cdot \sin B$

$\sin^2 A \cdot \cos B + \dfrac{1}{2} \sin 2A \cdot \sin B$

A-6. $\therefore \cos \left(\dfrac{\pi}{2} + \theta \right) = -\sin\theta$ [cos in II quadrant gives negative value]

$\sin (\theta - \pi) = \sin[-(\pi - \theta)] = -\sin(\pi - \theta)$
$= -\sin\theta$
$\sin(\pi + \theta) = -\sin\theta$ [sin function is –ve in III quadrant]

A-7. $x_1 = 8 \sin\theta$ and $x_2 = 6\cos\theta$ then
$x_1 + x_2 = 10 \sin(\theta + 37°)$
$(x_1 + x_2)_{max} = 10 [\sin(\theta + 37°)]_{max} = 10 \times 1 = 10$

Section (B) :

B-1. $\dfrac{dy}{dx} = 2x + 1$

B-2. $\dfrac{dy}{dx} = \sec^2 x - \text{cosec}^2 x$

B-3. $\dfrac{dy}{dx} = \cos x - \sin x$, $\dfrac{d^2y}{dx^2} = -\sin x - \cos x$

B-4. $\dfrac{dy}{dx} = \dfrac{1}{x} + e^x$, $\dfrac{d^2y}{dx^2} = -\dfrac{1}{x^2} + e^x$

Section (C) :

C-1. $\dfrac{d}{dx} e^x \ell nx = \ell n^x \dfrac{de^x}{dx} + e^x \dfrac{d\ell n^x}{dx}$

$e^x \ell nx. + \dfrac{e^x}{x}$

C-2. $\dfrac{d(\sin x \cos x)}{dx} = \sin x \dfrac{d(\cos x)}{dx} + \cos x \dfrac{d(\sin x)}{dx} = \cos^2 x - \sin^2 x$

Section (D) :

D-1. $y' = \dfrac{(3x-2)(2)-(2x+5)(3)}{(3x-2)^2} = \dfrac{-19}{(3x-2)^2}$

D-2. $\dfrac{dy}{dx} = \dfrac{d\left(\dfrac{\ell nx}{x}\right)}{dx} = \dfrac{x\times\dfrac{1}{x}-\ell nx.1}{x^2} = \dfrac{1-\ell nx}{x^2}$

D-3. $\sec^2x - \tan^2x = 1$

$\therefore \quad \dfrac{d(1)}{dx} = 0$

D-4. (a) $\dfrac{d}{dx}\,uv = uv^1 + u^1v$

$= 5 \times 2 + (-1)\,(-3) = 13$

(b) $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{vu^1 - uv^1}{v^2} = \dfrac{(-1)(-3)-(5)(2)}{(-1)^2} = -7$

(c) $\dfrac{d}{dx}\left(\dfrac{v}{u}\right) = \dfrac{uv^1 - vu^1}{u^2} = \dfrac{(5)(2)-(-1)(-3)}{(5)^2} = \dfrac{7}{25}$

(d) $\dfrac{d}{dx}\,(7v - 2u) = 7v^1 - 2u^1$

$= 7 \times 2 - 2\,(-3) = 20$

Section (E) :

E-1. $y = \sin 5x$

Let $5x = \theta$

$y = \sin\theta$

$\dfrac{dy}{dx} = \dfrac{dy}{d\theta}.\dfrac{d\theta}{dx}$

$\dfrac{dy}{d\theta} = \cos\theta \qquad \dfrac{d\theta}{dx} = 5$

$\therefore \quad \dfrac{d\theta}{dx} = 5\cos\theta \qquad \theta = 5x$

$\therefore \quad \dfrac{dy}{dx} = 5\cos 5x$

E-2. Let $u = (\omega x + \phi)$

$\left(\dfrac{dy}{dx}\right) = \dfrac{dy}{du}\times\dfrac{du}{dx}$

$= 2\cos(\omega x + \phi) \times \omega$

$= 2\omega\cos(\omega x + \phi)$ **Ans.**

E-3. $\dfrac{dy}{dx} = -27(4-3x)^8$

Section (F) :

F-1. $(x + y)^2 = 4$

$2(x + y)\left(1 + \dfrac{dy}{dx}\right) = 0$

$\therefore x + y \neq 0 \quad \Rightarrow \quad 1 + \dfrac{dy}{dx} = 0 \quad \Rightarrow \quad \dfrac{dy}{dx} = -1$

F-2. $x^2y + xy^2 = 6$

$x^2\left[\dfrac{dy}{dx}\right] + y[+2x] + y^2 + x\left[2y\ \dfrac{dy}{dx}\right] = 0$

$\dfrac{dy}{dx}[x^2 + 2xy] + 2xy + y^2 = 0$

$\therefore\ \dfrac{dy}{dx} = \dfrac{-2xy - y^2}{x^2 + 2xy}$

Section (G) :

G-1. $\dfrac{dA}{dt} = \dfrac{d(\pi r^2)}{dt} = \dfrac{\pi d(r^2)}{dt} = \dfrac{2\pi r dr}{dt}$

G-2. $\dfrac{d}{dt}S = \dfrac{d(4\pi r^2)}{dt} = 8\pi r\dfrac{dr}{dt}$

Section (H) :

H-1. $x = -t^2 + 4t + 4$ ....(i)

for maxima

$\dfrac{dx}{dt} = 0$

$-2t + 4 = 0$

$t = 2$

from equation (i) $\dfrac{d^2x}{dt^2} = -2 < 0$

x has maximum value at t = 2 sec. maximum value of x

$x_{max} = -(2)^2 + 4(2) + 4 = 8$

H-2. $y_{max} = 39,\ y_{min} = 38$

Section (I) :

I-1. $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

$\dfrac{dy}{dx} = 48(8x - 1)^2$

I-2. $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

$3\cos(3x + 1)$

I-3. $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

$12x^3.$

I-4. $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

$\dfrac{dy}{dx} = -\dfrac{1}{3}\sin\dfrac{x}{3}$

PART - II

Section (A) :

A-1. $y = x^2 - 2x + 1$

$\int y\,dx = \int\left(x^2 - 2x + 1\right)dx + c = \int x^2 dx - 2\int x\,dx + \int dx + c$

$= \dfrac{x^3}{3} - x^2 + x + c$

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A-2. $y = (x)^{1/3} + x^{-1/3}$

$\int y\,dx = \int x^{1/3}dx \; + \int x^{-1/3}dx$

$\dfrac{3x^{4/3}}{4} + \dfrac{3x^{2/3}}{2} + c$

A-3. $y = \sec^2 x$ $\qquad \int y\,dx = \tan x + c$

A-4. $y = \csc^2 x$ $\qquad \int y\,dx = -\cot x + c$

A-5. $y = \sec x \cdot \tan x = \int y\,dx \;\; \sec x + c$

A-6. $y = \dfrac{1}{3x}$ $\qquad \int y\,dx = \dfrac{1}{3}\int\dfrac{1}{x}dx + c = \dfrac{1}{3}\ell nx + c$

Section (B) :

B-1. $\int x\sin(2x^2)dx \quad$ Let $u = 2x^2 \qquad du = 4x\,dx$

$= \int \sin u\,\dfrac{du}{4}$

$= \dfrac{1}{4}\int \sin u\,du = -\dfrac{1}{4}\cos u + C$

B-2. $\int \sec 2t\,\tan 2t\,dt$

Let $u = 2t \qquad du = 2\,dt$

$= \int \sec u\,\tan u\,\dfrac{du}{2}$

$= \dfrac{1}{2}\sec u + C$

$= \dfrac{1}{2}\sec u + C$

B-3. $\int \dfrac{3}{(2-x)^2}\,dx = 3\int(2-x)^{-2}\,dx$

Let $u = 2 - x \qquad du = -\,dx$

$3\int u^{-2}(-dx)$

$= \dfrac{+3}{u} + C = \dfrac{3}{2-x} + C$

B-4. $\int \sin(8z - 5)\,dz$

Let $u = 8z - 5 \quad du = 8\,dz$

$\int \sin u\,\dfrac{du}{8} = \dfrac{1}{8}\,(-\cos u) + C$

$= -\dfrac{1}{8}\cos(8z - 5) + C$

Section (C) :

C-1. $\dfrac{\pi}{2}\displaystyle\int_{-4}^{-1} d\theta = \dfrac{\pi}{2}[\theta]_{-4}^{-1} = \dfrac{\pi}{2}\,[(-1) - (-4)] = \dfrac{3\pi}{2}$

C-2. $\displaystyle\int_{\sqrt{2}}^{5\sqrt{2}} r\,dr = \left[\dfrac{r^2}{2}\right]_{\sqrt{2}}^{5\sqrt{2}} = \dfrac{(5\sqrt{2})^2}{2} - \dfrac{(\sqrt{2})^2}{2} = 25 - 1 = 24$

C-3. $\displaystyle\int_0^1 e^x\,dx = \left[e^x\right]_0^1 = e - 1$

Section (D) :

D-1. $y = 2x$

$$\text{Area} = \int_0^b y\,dx = \int_0^b 2x\,dx = \left[x^2\right]_0^b = b^2 \text{ units}$$

D-2. $y = \dfrac{x}{2} + 1$

$$\int_0^b y\,dx = \int_0^b \dfrac{x}{2}dx + \int_0^b dx = \left[\dfrac{x^2}{4}\right]_0^b + \left[x\right]_0^b$$

$$= \dfrac{b^2}{4} + b$$

D-3. $\displaystyle\int_0^\pi y\,dx = \int_0^\pi \sin x\,dx = \left[-\cos x\right]_0^\pi$
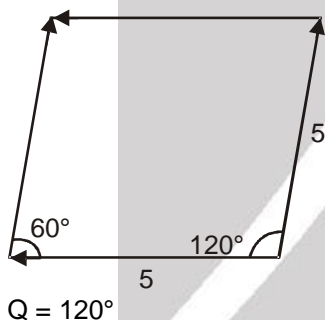
$$= [-\cos\pi + \cos 0] = 2$$

D-4. $\displaystyle\int_0^\pi \sin^2 x\,dx = \int_0^\pi \dfrac{1-\cos 2x}{2}\,dx$

$$= \int_0^\pi \dfrac{1}{2}dx - \int_0^\pi \dfrac{\cos 2x}{2}\,dx$$

$$= \dfrac{\pi}{2} - \dfrac{1}{4}\left[\sin 2x\right]_0^\pi = \dfrac{\pi}{2} - 0 = \dfrac{\pi}{2}$$

PART - III

Section (A) :

A-1. (i) 105⁰, (ii) 150⁰, (iii) 105⁰.

A-2.

Q = 120°

A-3. $V_R = 5\,\hat{j}$ m/s $= -5\,\hat{j}$ m/s.

Section (B) :

B-1.

$\left|\vec{OP}\right|$ = 30 m

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B-2. $\vec{A} = 30\,\hat{i}$

$\vec{B} = 40\,\hat{j}$

Resultant vector

40

ϕ

30
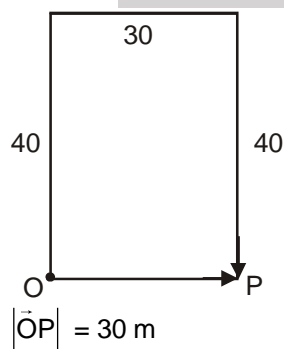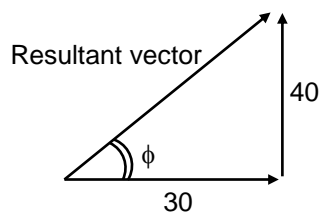
$\hat{i} = \vec{A} + \vec{B} = 30\,\hat{i} + 40\,\hat{j}$

$\tan \phi = 4/3 = 53°$

B-3.

$\left|\vec{C}\right| = \sqrt{a^2 + b^2 + 2ab\cos\theta}$ $\left|\vec{a}^1\right| = \left|\vec{b}\right|$

$\left|\vec{C}\right| = \sqrt{a^2 + b^2 + 2ab\cos\theta}$ $\left|\vec{b}^1\right| = \left|\vec{a}\right|$

$\therefore \left|\vec{c}^1\right| = \left|\vec{c}\right|$
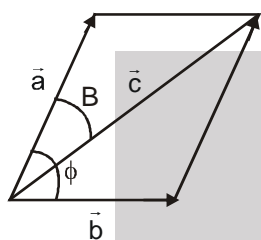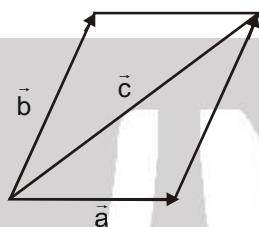
B-4. Can not be zero

B-5.

10   6

C
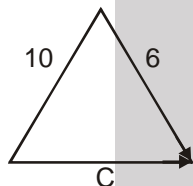
Using triangular inequality

$C > 10 - 6$

$C < 10 + 6$

$\therefore \qquad 4 < c < 16$

B-6. Resultant of vectors forming closed polygon is zero vector.

B-7. $\left|\vec{A} + \vec{B}\right| = \sqrt{A^2 + \mathbf{B}^2 + 2AB \cdot \cos\theta}$

$\cos\theta$ is min. when $\theta = \pi$

B-9. $\vec{C} = \vec{A} + \vec{B}$

$|\vec{A}| = 12 \qquad |\vec{C}| = 13$

$\vec{C}$ $\vec{A}$

$\vec{B}$

$|\vec{B}| = 5$

They for ether form pythagorian Triplet with $Q_{AB} = 90$

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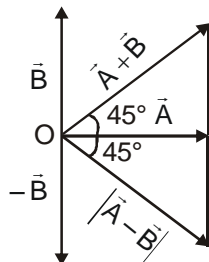
ADVMT - 6
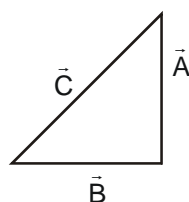
B-10. $\vec{P} + \vec{Q} = \vec{P} - \vec{Q}$

$2\vec{Q} = 0$

$\Rightarrow \quad \vec{Q} = 0$

B-11.

Section (C) :

C-1. $\sqrt{(3)^2 + (2)^2 + (1)^2} = \sqrt{9+4+1} = \sqrt{14}$ unit.

C-2. $x = 25 \cos^2 10° = -25 \cos 30°$ [210° = 180° + 30°]

$y = 25 \sin^2 10° = -25 \sin 30°$

C-3. $\left|\vec{V}\right| = 60$ km/h

$\left|\vec{V}\right| = \sqrt{30^2 + x^2} = 60$

$30^2 + x^2 = 60°$

$x^2 = 3600 - 900$

$x^2 = 2700$

$x^2 = 900 \times 3$

$x = 30\sqrt{3}$

C-4. $\vec{A} = 0.5\hat{i} + 0.8\hat{j} + c\hat{k}$ is a unit vector

$0.5^2 + 0.8^2 + c^2 = 1^2$

$\therefore \quad c = \pm \dfrac{\sqrt{11}}{10}$

C-5. $\vec{A} = 2\hat{i} + 2\hat{j}$

$\cos\theta = \dfrac{\vec{A} \cdot \vec{B}}{\left|\vec{A}\right|\left|\vec{B}\right|} = \dfrac{2 + 2\sqrt{3}}{\left(2\sqrt{2}\right)(2)}$

$\vec{B} = \hat{i} + \sqrt{3}\,\hat{j}$

$= \dfrac{2\left(1+\sqrt{3}\right)}{4\left(\sqrt{2}\right)} = \dfrac{1+\sqrt{3}}{2\sqrt{2}} \quad \Rightarrow Q = 15°$

C-6. $\vec{F} = 2\hat{i} - 3\hat{j}$ N

C-7. A(1, 1, –1)   B(2, –3, 4)

$\overrightarrow{AB} = \vec{B} - \vec{A} = (2, -3, 4) - (1, -1, 1) = (1, -4, 5)$

$\therefore \quad \overrightarrow{AB} = \hat{i} - 4\hat{j} + 5\hat{k}$

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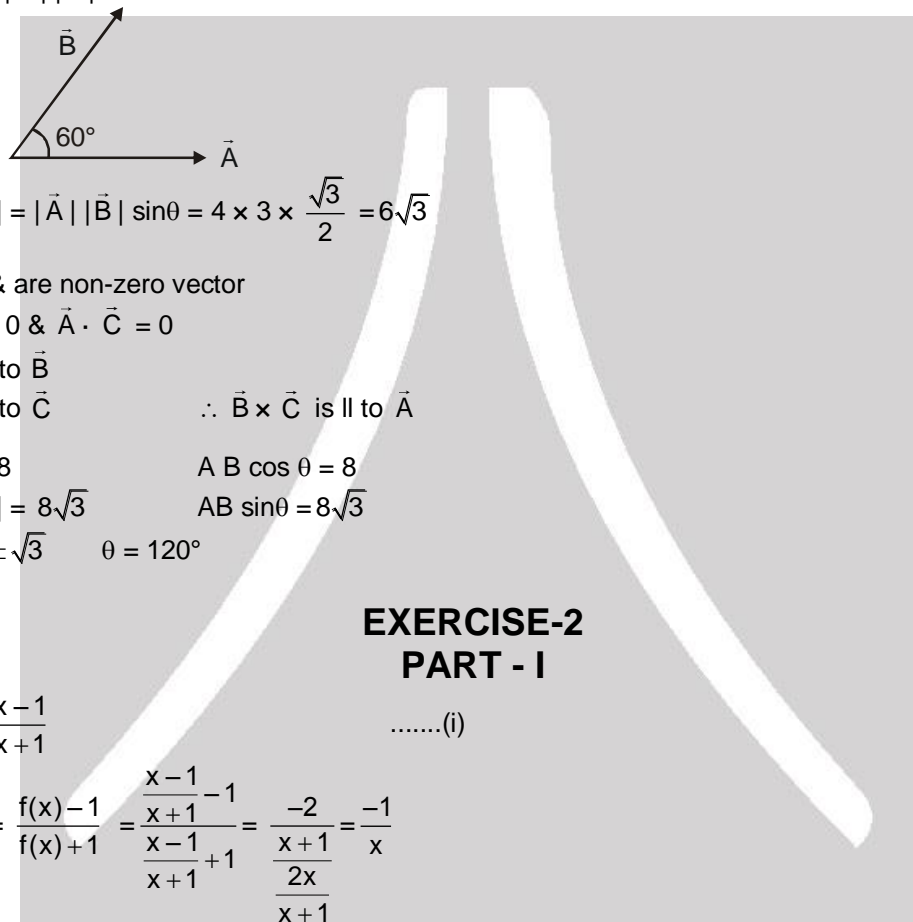
ADVMT - 7

Section (D) :

D-1. $\vec{A} = \hat{i} + \hat{j} + \hat{k}$ $\vec{B} = 2\hat{i} + \hat{j}$

(a) $\vec{A} \cdot \vec{B} = (1)(2) + (1)(1) + (1)(0) = 2 + 1 = 3$

$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{vmatrix} = \hat{i}(0 - 1) - \hat{j}(0 - 2) + \hat{k}(1 - 2)$

$\vec{A} \times \vec{B} = -\hat{i} + 2\hat{j} - \hat{k}$

D-2. $|\vec{A}| = 4 \quad |\vec{B}| = 3$

$\theta = 60°$

$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos 60° = 4 \times 3 \times 1/2 = 6$

$|\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}| \sin\theta = 4 \times 3 \times \dfrac{\sqrt{3}}{2} = 6\sqrt{3}$

D-3. $\vec{A}$, $\vec{B}$ & are non-zero vector

$\vec{A} \cdot \vec{B} = 0$ & $\vec{A} \cdot \vec{C} = 0$

$\vec{A}$ is $\perp$ to $\vec{B}$

$\vec{A}$ is $\perp$ to $\vec{C}$ $\therefore$ $\vec{B} \times \vec{C}$ is ll to $\vec{A}$

D-4. $\vec{A} \cdot \vec{B} = 8$ $A\,B \cos\theta = 8$

$|\vec{A} \times \vec{B}| = 8\sqrt{3}$ $AB \sin\theta = 8\sqrt{3}$

$\tan\theta = \pm\sqrt{3}$ $\theta = 120°$

EXERCISE-2
PART - I

1. $f(x) = \dfrac{x-1}{x+1}$ ....(i)

$f\{f(x)\} = \dfrac{f(x)-1}{f(x)+1} = \dfrac{\dfrac{x-1}{x+1}-1}{\dfrac{x-1}{x+1}+1} = \dfrac{\dfrac{-2}{x+1}}{\dfrac{2x}{x+1}} = \dfrac{-1}{x}$

2. $f(y) = f[f(x)] = \dfrac{2\left[\dfrac{2x-3}{3x-2}\right]-3}{3\left[\dfrac{2x-3}{3x-2}\right]-2} = x$

3. $120° + \theta + \theta = 180°$

$\Rightarrow$ $\theta = 30°$

$\dfrac{\sin 120}{10} = \dfrac{\sin \angle A}{a} = \dfrac{\sin \angle C}{c}$

side a = side c = $\dfrac{10\sqrt{3}}{3}$ m

5. (A) $\dfrac{3}{5}+\dfrac{4}{5}=\dfrac{4}{5}+\dfrac{3}{5}$ 　　　　(B) $\dfrac{3}{5}-\dfrac{4}{5}=\dfrac{3}{5}-\dfrac{4}{5}$

　　(C) $\dfrac{3}{4}+1\neq\dfrac{4}{3}-1$ 　　　　(D) $\dfrac{3}{4}\times\dfrac{4}{3}=1$

6. $R^2 = 2A^2(1+\cos\theta) = 2A^2\left(1+2\cos^2\dfrac{\theta}{2}-1\right) = 2^2A^2\cos^2\dfrac{\theta}{2}$

$R = 2A\cos\dfrac{\theta}{2}$

7. $y = \ln x^2 + \sin x$

$\dfrac{dy}{dx} = \dfrac{d(\ln x^2)}{dx}+\dfrac{d(\sin x)}{dx}$

$= \dfrac{1}{x^2}\,\dfrac{d}{dx}(x^2) + \cos x$

$= \dfrac{1}{x^2}\,.\,2x + \cos x$

$= \dfrac{2}{x} + \cos x$

$\Rightarrow \quad \dfrac{d^2y}{dx^2} = \dfrac{d(\frac{2}{x})}{dx} + \dfrac{d(\cos x)}{dx} = \dfrac{-2}{x^2} - \sin x$

8. $\dfrac{dy}{dx} = \dfrac{d(x^{1/7})}{dx}+\dfrac{d(\tan x)}{dx} = \dfrac{x^{-\frac{6}{7}}}{7} + \sec^2 x$

$\dfrac{d^2y}{dx^2} = \dfrac{1}{7}\dfrac{d(x^{-6/7})}{dx}+\dfrac{d(\sec^2 x)}{dx} = \dfrac{-6}{49}x^{-13/7} + 2\sec x\,(\sec x\,\tan x)$

$= \dfrac{-6}{49}x^{-13/7} + 2\tan x\,\sec^2 x$

9. $y = \left(x+\dfrac{1}{x}\right)\left(x+\dfrac{1}{x}+1\right)$

$\dfrac{dy}{dx} = \left(x+\dfrac{1}{x}\right)\dfrac{d(x-\frac{1}{x}+1)}{dx}+\left(x-\dfrac{1}{x}+1\right)\dfrac{d\left(x+\frac{1}{x}\right)}{dx}$

$\left(x+\dfrac{1}{x}\right) + \left[\dfrac{dx}{dx}-\dfrac{d\ (\frac{1}{x})}{dx}+\dfrac{d\ (1)}{dx}\right]+\left(x-\dfrac{1}{x}+1\right)\left[\dfrac{dx}{dx}+\dfrac{d(\frac{1}{x})}{dx}\right]$

$= \left(x+\dfrac{1}{x}\right)\left(1+\dfrac{1}{x^2}\right) +\left(x-\dfrac{1}{x}+1\right)\left(1-\dfrac{1}{x^2}\right)$

$= x + \dfrac{1}{x} + \dfrac{1}{x} + \dfrac{1}{x^3} + x - \dfrac{1}{x} - \dfrac{1}{x} + \dfrac{1}{x^3} + 1 - \dfrac{1}{x^2} = \dfrac{2}{x^3} + 2x - \dfrac{1}{x^2} + 1$

10. $r = (1+\sec\theta)\sin\theta$

$r = \sin\theta + \sec\theta\sin\theta$

$r = \sin\theta + \tan\theta$

$\dfrac{dr}{d\theta} = \cos\theta + \sec^2\theta$

11.

$$\frac{dy}{dx} = \frac{(1+\cot x)\dfrac{d}{dx}(\cot x) - \cot x \dfrac{d}{dx}(1+\cot x)}{(1+\cot x)^2}$$

$$= \frac{(1+\cot x)(-\csc^2 x) - \cot x(-\csc^2 x)}{(1+\cot x)^2} = \frac{-\csc^2 x}{(1+\cot x)^2}$$

12.

$$\frac{dy}{dx} = \frac{\tan x \dfrac{d}{dx}(\ell nx + e^x) - (\ell nx + e^x)\dfrac{d}{dx}\tan x}{(\tan x)^2}$$

$$= \frac{\tan x\left(\dfrac{1}{x} + e^x\right) - \left(\ell nx + e^x\right)(\sec^2 x)}{(\tan x)^2}$$

13.

$$\frac{d}{dx}(\sin^3 x) + \frac{d}{dx}(\sin 3x)$$

$$= \frac{d\sin^2 x}{d\sin x} \times \frac{d\sin x}{dx} + \frac{d\sin 3x}{d(3x)} \times \frac{d(3x)}{dx} = 3\sin^2 x \cos x + \frac{\cos 3x}{3}$$

$$= 3\sin^2 x \cos x + 3\cos 3x$$

14.

$$\frac{dy}{dx} = \frac{d}{dx}[\sin^2(x^2 + 1)]$$

$$= \frac{d\sin^2(x^2 + 1)}{d\sin(x^2 + 1)} \times \frac{d\sin(x^2 + 1)}{d(x^2 + 1)} \times \frac{d(x^2 + 1)}{dx}$$

$$= 4x\sin(x^2 + 1)\cos(x^2 + 1)$$

15.

$$\frac{dq}{dr} = \frac{d}{dr}(2r - r^2)^{1/2}$$

$$= \frac{d(2r - r^2)^{1/2}}{d(2r - r^2)} \times \frac{d(2r - r^2)}{dr}$$

$$= \frac{1}{2}(2r - e^2)^{-1/2}(2 - 2r) = \frac{1 - r}{\sqrt{2r - r^2}}$$

16. $x^3 + y^3 = 18xy$

$$\frac{d(x^3 + y^3)}{dx} = \frac{d(18xy)}{dx}$$

$$\frac{dx^3}{dx} + \frac{dy^3}{dx} = 18x.\frac{dy}{dx} + y.\frac{d(18x)}{dx}$$

$$3x^2 + 3y^2.\frac{dy}{dx} = 18x.\frac{dy}{dx} + y.18$$

$$3x^2 + 3y^2.\frac{dy}{dx} = 18x.\frac{dy}{dx} + 18y$$

$$3y^2\frac{dy}{dx} - 18x.\frac{dy}{dx} = 18y - 3x^2$$

$$\frac{dy}{dx}(3y^2 - 18x) = 18y - 3x^2$$

$$\frac{dy}{dx} = \frac{(18y - 3x^2)}{(3y^2 - 18x)}$$

17. $V = \pi r^2 h.$

$$\frac{dv}{dt} = \frac{d\pi r^2 h}{dt} = \pi \frac{dr^2 h}{dt}$$

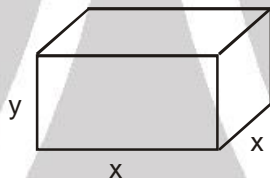$$= \pi \left\{ h \frac{dr^2}{dt} + r^2 \frac{dh}{dt} \right\}$$

$$= \pi \left\{ 2rh \frac{dr}{dt} + r^2 \frac{dh}{dt} \right\}$$

$$= \pi \{ 2rh \cdot 5 + r^2 \cdot 5 \} \left\{ \frac{dh}{dt} = 5 \, m/s \, \& \, \frac{dr}{dt} = 5 \, m/s \right\}$$

$$= \pi \{ 10rh + 5r^2 \}$$

18. $x + y = 60$
$x = 60 - y$
$xy = (60 - y) \, y$
$f(y) = (60 - y) \, y$
for maximum
$f'(y) = 60 - 2y = 0$
$y = 30$
so $x = 30$ & $y = 30$

19. Let the dimensions of the tank be x and y area of the open tank $= x^2 + 4xy$.
Again x & y are related to surface area of this tank which is equal to 40 m²

$x^2 + 4xy = 40$

$y = \dfrac{40 - x^2}{4x}$

Volume $v = x^2 \left(\dfrac{40 - x^2}{4x} \right) = \dfrac{40x - x^3}{4}$

for maximum volume $v'(x) = \dfrac{40 - 3x^2}{4} = 0$

$x = \sqrt{\dfrac{40}{3}}$ and $v''(x) = -\dfrac{3x}{2}$

$v'' \sqrt{\dfrac{40}{3}} = -\dfrac{3}{2} \sqrt{\dfrac{40}{3}} < 0$

So volume is maximum at $x = \sqrt{\dfrac{40}{3}}$ m

PART - II

1. $= \int (x^{-2} + x^{-3}) \, dx$

$= \int x^{-2} dx + \int x^{-3} dx$

$= \dfrac{x^{-2+1}}{-2+1} + \dfrac{x^{-3+1}}{-3+1} + C$

$= -x^{-1} - \dfrac{1}{2} x^{-2} + C,$

2. $\int (1 - \cot^2 x)dx = \int 1 - (\cosec^2 x - 1)dx$

$= \int (2 - \cosec^2 x)dx$

$= \int 2dx - \int \cosec^2 x\, dx$

$= 2x + \cot x + C$

3. $\int \cos\theta\ (\tan\theta + \sec\theta)\ d\theta$

$= \int \cos\theta\ \tan\theta\ d\theta + \int \cos\theta\ \sec\theta\ d\theta$

$= \int \cos\theta \dfrac{\sin\theta}{\cos\theta}d\theta + \int d\theta$

$= -\cos\theta + \theta + C$

4. $\int 1\,2\ (y^4 + 4y^2 + 1)^2\ (y^3 + 2y)\ dy$

By Substituting $u = y^4 + 4y^2 + 1$

$du = dy = \dfrac{d(y^4 + 4y^2 + 1)}{dy}\ (4y^3 + 8y)\ dy = 4\ (y^3 + 2y)\ dy$

Then, $\int 12\ (y^4 + 4y^2 + 1)^2(y^3 + 2y)\ dy = \int 3u^2 du$

$= 3\int u^2 du = \dfrac{3u^3}{3} + c = (y^4 + 4y^2 + 1)^3 + C$

5. $\int \dfrac{dx}{\sqrt{5x + 8}}$.

By Substituting $5x + 8 = u$,

$\dfrac{d(5x + 8)}{dx} = \dfrac{du}{dx}$

$5 = \dfrac{du}{dx}$

$5dx = \left(\dfrac{du}{dx}\right)dx$

$dx = \dfrac{du}{5}$

then $\int \dfrac{du}{5\sqrt{u}} = 1/5 \int u^{-\frac{1}{2}}\ du = \dfrac{1}{5}\dfrac{u^{1/2}}{1/2} + C$

$= \dfrac{2}{5}\sqrt{u} + C = \dfrac{2}{5}\ \sqrt{5x + 8} + C$

6. Let $u = 3 - 2s \Rightarrow du = -2\ ds$

$\int \sqrt{u}\ \left(\dfrac{-du}{2}\right) = \dfrac{-1}{2\times 3/2}\ u^{3/2} + C = \dfrac{-1}{3}\ (3 - 2s)^{3/2} + C$

7. $\int \sec^2(3x + 2)dx$

Let $u = 3x + 2 \qquad du = 3dx$

$= \int \sec^2 u \dfrac{du}{3}$

$= \dfrac{1}{3}\tan u + C = \dfrac{1}{3}\tan(3x + 2) + C$

8. $\int \csc\left(\dfrac{\upsilon - \pi}{2}\right)\cot\left(\dfrac{\upsilon - \pi}{2}\right)\, d\upsilon$

Let u = $\dfrac{\upsilon - \pi}{2}$

du = $\dfrac{dv}{2}$

= $\int \csc u \cot u\, (2du)$

= $-2\csc u + C$

= $-2\csc\left(\dfrac{\upsilon - \pi}{2}\right) + C$

9. $\int \dfrac{6\cos t}{(2 + \sin t)^3}$,

Substituting 2 + sin t = u

$\quad\quad\cos t = \dfrac{du}{dt}$

$\quad\quad\cos t\ dt = du$

Then $\int \dfrac{6du}{u^3} = 6\int \dfrac{du}{u^3}$

$\quad\quad = \dfrac{6u^{-3+1}}{-3+1} + C$

$\quad\quad = -3\ (2 + \sin t)^{-2} + C$

10. $\left[\dfrac{\theta^2}{2}\right]_{\pi}^{2\pi} = \dfrac{4\pi^2}{2} - \dfrac{\pi^2}{2} = \dfrac{3\pi^2}{2}$ **Ans.**

11. $= \left[\dfrac{x^3}{3}\right]_0^{3\sqrt{7}}$

$= \dfrac{\left((7)^{1/3}\right)^3}{3} - 0 = \dfrac{7}{3}$

12. Substituting $x^2 = u$,

or $\quad 2x\ dx = du$

$\quad\quad x\ dx = \dfrac{du}{2}$.

Now, changing the limit for

$\quad\quad x = 0,\quad\quad\quad u = 0$

$\quad\quad x = \sqrt{\pi}\ ,\quad\quad u = 2$

on $\displaystyle\int_{u=0}^{u=\pi}\sin u\, du = \left[-\cos u\right]_0^{\pi} = -\cos \pi + \cos 0 = 2$

13. $\left[\dfrac{1}{3}\ln(3x + 2)\right]_0^1$

$= \dfrac{1}{3}(\ell n\, 5 - \ell n 2)$

$= \dfrac{1}{3}\ell n\ \dfrac{5}{2} = \ell n\ \left(\dfrac{5}{2}\right)^{1/3}$

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ADVMT - 13

14. Area of the region between the given curve & x axis on the interval [0, b]
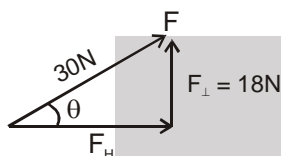
$$= \int_0^b 3x^2 dx$$

$$= \left[\frac{3x^3}{3}\right]_0^b = \left[x^3\right]_0^b = b^3 - 0 = b^3$$

PART - III

1. $\vec{B} = \lambda\vec{A} = -4 \times 3$ N-E

= 12 S-W

No it does not represent the same physical quantity.

2.

Given that $F_\perp = 18$

From figure $F_H = \sqrt{F^2 - F_1^2} = \sqrt{30^2 - 18^2}$

$= \sqrt{576} = 24$N

From figure $\tan\theta = \dfrac{F_1}{F_H} = \dfrac{18}{24}$

$\tan\theta = 3/4 \quad\Rightarrow\quad \theta = 37°$

3. Let $\vec{P}$ & $\vec{Q}$ are two vector

If

P – Q = 10 unit .....(1)

If

$\sqrt{P^2 + Q^2} = 50$ unit

$P^2 + Q^2 = 50^2$ .....(2)

from equation (1)

$(10 + Q)^2 + Q^2 = 50^2$

$2Q^2 + 20Q + 100 = 2500$

$2Q^2 + 20Q = 2400$

$Q^2 + 10Q - 1200 = 0$

$(Q + 40) (Q - 30) = 0$

Q = 30

So, from equation (1)

P = 10 + Q

= 10 + 30 = 40 unit

4. $\tan\alpha = \dfrac{\beta\sin\theta}{A + B\cos\theta}$

$\tan\alpha = \dfrac{6\sin 90}{8 + 6\cos 90} = \dfrac{1}{8} = \dfrac{3}{4}$

$\tan\alpha = 3/4$

$\alpha = 37$ Ans.

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ADVMT - 14

5. $\overrightarrow{OA} = r\,\hat{j}$

$\overrightarrow{OC} = r\,\hat{i}$

$\overrightarrow{OB} = r\,(\cos 45\,\hat{i} + \sin 45\,\hat{j})$

$\overrightarrow{OB} = \dfrac{r}{\sqrt{2}}\,\hat{i} + \dfrac{r}{\sqrt{2}}\,\hat{j}$

$\vec{R} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$

$= r\,\hat{j} + \dfrac{r}{\sqrt{2}}\,\hat{i} + \dfrac{r}{\sqrt{2}}\,\hat{j} + r\,\hat{i}$

$\left|\vec{R}\right| = \left|\left(r + \dfrac{r}{\sqrt{2}}\right)\hat{i} + \left(r + \dfrac{r}{\sqrt{2}}\right)\hat{j}\right| = r\left(1 + \sqrt{2}\right)$

7. Given that

$\vec{A} = 4\,\hat{i} + 6\,\hat{j}$ .... (1)

$\vec{A} + \vec{B} = 10\,\hat{i} + 9\,\hat{j}$ .... (2)

from equation (2)

$\vec{B} = 10\,\hat{i} + 9\,\hat{j} - \vec{A}$

$= 10\,\hat{i} + 9\,\hat{j} - (4\,\hat{i} + 6\,\hat{j}) = 6\,\hat{i} + 3\,\hat{j}$
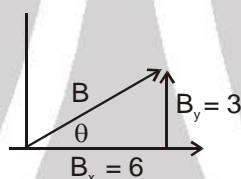
From figure $\tan\theta = \dfrac{B_y}{B_x} = \dfrac{3}{6} = \tan^{-1}\left(\dfrac{1}{2}\right)$

9. Angle between two vectors can never be greater than 180°

on increasing the θ, the magnitude of resultant vectors decreases.

10. Initial velocity = $50\,\hat{j}$

Final velocity = $-50\,\hat{j}$

1 change = $50\sqrt{2}$
along south west

11. Sum of any 3 sides should be greater than fourth side.

12. $a + b > \left|\vec{a} + \vec{b}\right| > a - b$

13. $|\vec{A} + \vec{B}| = |\vec{A}| = |\vec{B}|$

Case -I Either $|\vec{A}| = |\vec{B}| = 0$ (zero vectors)

Case - II $|A| = |B| \neq 0$

$|\vec{A} + \vec{B}| = A^2 + B^2 + AB\cos\theta$

$|A| = |B|$

$= 2A^2 + 2A^2\cos\theta$

$= 2A^2(1 + \cos\theta)$

= 2A (2cos^2 θ/2)

= 4A cos^2 θ/2

Now |A + B| = A

$\cos\left|\dfrac{\theta}{2}\right| = \dfrac{1}{2}$ $\dfrac{\theta}{2} = 60°$

θ = 120°

14. $\vec{a}$ 45° $\vec{b}$ By vector translation

45°

$\vec{c}$

135° $\vec{a}$

$\vec{c}$ 135° $\vec{a}$

$\vec{b}$ $\vec{b}$

$\vec{c}$

$\Downarrow$

$\vec{c}$

$\vec{b}$

∴ 90º, 135º, 135º

15.

$\vec{a}$

60°

60°

$\vec{b}$
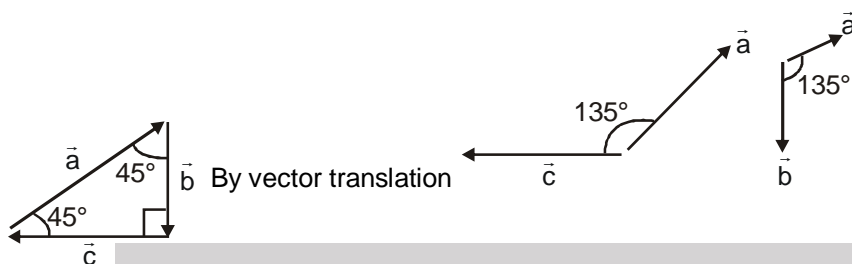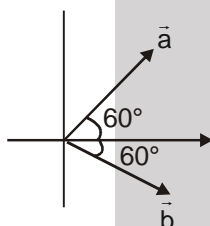
Only horizontal along + x-axis

⇒   2cos 60° + 2 cos 60° = 2

16.

θ θ θ θ θ θ
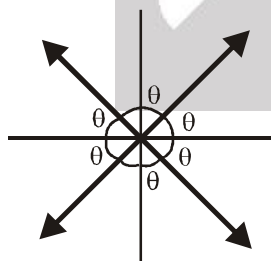
Coplanar all in a single plane (xy plane)

6θ = 360°

θ = 60°

Check now every component cancels out net = 0

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17.
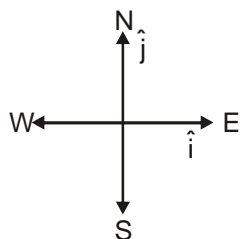
$\vec{A} \to -\hat{k}$

$\vec{B} \to +\hat{i}$

$\vec{A} \times \vec{B} = -\hat{k} \times \hat{i} = -\hat{j} \Rightarrow$ south

18. A vector is represented by sum of its component vectors.

i.e., $\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$

so, from given options only option (A), (B), (C) are correct.

19. $a^2 + b^2 + 2ab\cos\theta = a^2 + 4b^2 - 4ab\cos\theta$

or $\cos\theta = \dfrac{b}{2a} < 1$

∴ $b < 2a$

EXERCISE-3
PART - I

1. (A) $\int \sec x \tan x\, dx = \sec x + C$

(B) $\int \cos ec\, kx \cot kx\, dx = \dfrac{-\cos ec\, kx}{k} + C$

(C) $\int \cos ec^2\, kx\, dx = -\dfrac{\cot kx}{k} + C$

(D) $\int \cos kx\, dx = \dfrac{\sin kx}{k} + C$

2. (A) $|\vec{A} + \vec{B}| = A^2 + B^2 + 2AB\cos\theta$

$A^2 = A^2 + A^2 + 2A^2 \cos\theta$ $\cos\theta = -\dfrac{1}{2}, \theta = 120$

(B) $F_1 \sim F_2 \le R \le F_1 + F_2$
Here $F_1 \sim F_2 = 4$ and $F_1 + F_2 = 12$

(C) $\cos\theta = \dfrac{\vec{A} \cdot \vec{B}}{|A||B|} = \dfrac{0}{2\sqrt{2} \times 3} = 0$ $\theta = 90^0$

(D) $\vec{A} + \vec{B} = 2\hat{i} + \hat{j} + 3\hat{k}$ $|\vec{A} + \vec{B}| = \sqrt{2^2 + 1 + 3^2} = \sqrt{14}$

PART - II

1. $x = t^3 - 3t^2 + 12t + 20$

$v = \dfrac{dx}{dt} = 3t^2 - 6t + 12$

$t = 0 \Rightarrow v = 12$ m/s

2. $a = \dfrac{dv}{dt} = 6t - 6$

$t = 0 \Rightarrow a = -6$ m/s²

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ADVMT - 17

3. $a = 0 \Rightarrow t = 1$ sec.

$v = 3t^2 - 6t + 12 = 9$ m/s

4. $F_{net} = \vec{F}_1 + \vec{F}_2 = 2\hat{i} + 5\hat{j} + 4\hat{k}$

5. $\cos\theta = \dfrac{\vec{F}_1 \cdot \vec{F}_2}{|\vec{F}_1||\vec{F}_2|} = \left(\dfrac{3}{5\sqrt{2}}\right)$

6. $F_1 \cos\theta = \dfrac{\vec{F}_1 \cdot \vec{F}_2}{|\vec{F}_2|} = \dfrac{6}{5}$

PART - III

1. $|a| - |b| \le |\vec{a} + \vec{b}| \le |a| + |b|$

2. $R = \sqrt{A_x^2 + B_y^2} = \sqrt{g^2 + 6^2} = 10$

3. Based on theory

4. Based on theory

5. Angle between two vectors given by.

$\cos\theta = \dfrac{\vec{A}.\vec{B}}{AB}$

so $\cos\theta = \dfrac{(\hat{i} + \hat{j}).\hat{k}}{|(\hat{i} + \hat{j})||\hat{k}|} = \dfrac{0}{\sqrt{2} \times 1} = 0$

$\cos\theta = 0 \qquad \theta = \dfrac{\pi}{2}$

6. Based on theory.

PART - IV

1. **(i)** if $f(x) = e^{-x}$
then $f'(x) = -e^{-x}$
so $f(x) = -f'(x)$
(ii) if $f(x) = e^x$ then $f'(x) = e^{-x}$
so $f(x) = f'(x)$
(iii) Based on theory
(iv) Based on theory
(v) Based on theory

PART - V

1. $\vec{A} = A\hat{i} \qquad \vec{B} = A\hat{j} \qquad \vec{C} = A\hat{k}$

$\vec{A} + \vec{B} + \vec{C} = A\hat{i} + A\hat{j} + A\hat{k}$

$|\vec{A} + \vec{B} + \vec{C}| = \sqrt{A^2 + A^2 + A^2} = \sqrt{3}\,A$

2. Given $\vec{A} = 3\hat{i} + 4\hat{j} \qquad \vec{B} = 7\hat{j} + 24\hat{j}$

Let $\vec{C} = |\vec{C}|\,\hat{C}$

Given that $|\vec{C}| = |\vec{B}| = \sqrt{7^2 + (24)^2} = 25$

and $\hat{C} = \hat{A} = \dfrac{3\hat{i} + 4\hat{j}}{5}$

$\vec{C} = \dfrac{25 \times (3\hat{i} + 4\hat{j})}{5} \qquad \vec{C} = 15\hat{i} + 20\hat{j}$

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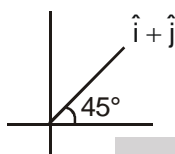
3. If $|\vec{A}\vec{B}|$

then angle between $\vec{A}$ and $\vec{B}$ is equal to zero

So, $\vec{A} \times \vec{B} = AB\sin\theta\, n = 0$

4. Area of parallelogram $= |\vec{A}\times\vec{B}| = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & 0 \\ 2 & 0 & -4 \end{vmatrix} = |-8\hat{i}+12\hat{j}-4\hat{k}| = \sqrt{224}$

5. Perpendicular

6.

7. $\vec{A} = a\hat{i}$ and $\vec{B} = a\cos\omega t\,\hat{i} + a\sin\omega t\,\hat{j}$

$|\vec{A}+\vec{B}| = \sqrt{3}\,|\vec{A}-\vec{B}|$

$\sqrt{(a+a\cos\omega t)^2 + (a\sin\omega t)^2} = \sqrt{3}\,\sqrt{(a-a\cos\omega t)^2 + (a\sin\omega t)^2}$

$\Rightarrow \qquad 2\cos\dfrac{\omega t}{2} = \pm\sqrt{3}\times 2\sin\dfrac{\omega t}{2}$

$\tan\dfrac{\omega t}{2} = \pm\dfrac{1}{\sqrt{3}}$

$\dfrac{\omega t}{2} = n\pi \pm \dfrac{\pi}{6}$

$\dfrac{\pi}{12}t = n\pi \pm \dfrac{\pi}{6}$

$t = (12n \pm 2)s$

$= 2s, 10s, 14s \ldots\ldots\ldots\ldots$

HIGH LEVEL PROBLEMS (HLP)
PART - I

1. $g'(x) = \dfrac{(x+0.5)\,\dfrac{d}{dx}(x^2-4) - (x^2-4)\dfrac{d}{dx}(x+0.5)}{(x+0.5)^2}$

$= \dfrac{(x+.5)(2x) - (x^2-4)}{(x+0.5)^2}$

$= \dfrac{2x^2 + x - x^2 + 4}{(x+0.5)^2} = \dfrac{x^2 + x + 4}{(x+0.5)^2}$

2. (a) $\dfrac{d}{dx}(uv) = uv' + vu'$

at $x = 1$ $\qquad = u(1)\,v'(1) + v(1)\,u'(1)$
$\qquad\qquad\qquad = 2x\,(-1) + 5\,(0)$
$\qquad\qquad\qquad = -2$

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(b) $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{vu'-uv'}{v^2}$

at x = 1 $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v(1)\ u'(1)-u\ (1)v'(1)}{[v(1)]^2}$

$= \dfrac{5(0)-(2)(-1)}{(5)^2} = \dfrac{2}{25}$

(c) $\dfrac{d}{dx}\left(\dfrac{v}{u}\right) = \dfrac{uv'-vu'}{u^2} = \dfrac{uv'(1)-v(1)u'(1)}{[u(1)]^2}$

(d) $\dfrac{d}{dx}(7v-2u) = 7v'-2u'$

at x = 1

$= 7\,v'(1) - 2u'\,(1)$

$7(-1) - 2(0) = -7$

3. $S = \dfrac{1+\cosec\ t}{1-\cosec\ t}$

$\dfrac{ds}{dt} = \dfrac{(1-\cosec t)(-\cosec t\cot t)-(1+\cosec t)(+\cosec t\cot t)}{(1-\cosec t)^2}$

$= \dfrac{-2\cosec t\cot t}{(1-\cosec t)^2}$

4. $S = \dfrac{\sin t}{1-\cos t}$

$\dfrac{ds}{dt} = \dfrac{(1-\cos t)\cos t-\sin t(\sin t)}{(1-\cos t)^2}$

$= \dfrac{\cos t - (\cos^2 t + \sin^2 t)}{(1-\cos\ t)^2} = \dfrac{\cos t-1}{(1-\cos t)^2} = \dfrac{1}{\cos t-1}$

5. With u = sin x, y = u³ : $\dfrac{dy}{dx} = \dfrac{dy}{du}\,\dfrac{du}{dx} = 3u^2\cos x = 3\sin^2 x\,(\cos x)$

6. With u = cosx

y = 5u⁻⁴

$\dfrac{dy}{dx} = \dfrac{dy}{du}\times\dfrac{du}{dx}$

$= 5\,(-4)\,u^{-5}\,(-\sin x)$

$\dfrac{dy}{dx} = 20\sin x\cos^{-5}x$

7. r = (csc θ + cot θ)⁻¹

r = $\dfrac{1}{\cosec\theta + \cot\theta}$

$\dfrac{dr}{d\theta} = \dfrac{(\cot\theta+\cosec\theta)\ (0)-1\ (-\cosec^2\theta\ -\cosec\theta\ \cot\theta)}{(\cot\theta+\cosec\theta)^2}$

$\dfrac{dr}{d\theta} = \dfrac{\cosec\theta}{\cot\theta+\cosec\theta}$

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8. $r = - (\sec\theta + \tan\theta)^{-1}$

$r = \dfrac{-1}{\sec\theta + \tan\theta}$

$\dfrac{dr}{d\theta} = \dfrac{1\ (\sec\theta\ \ \tan\theta + \sec^2\theta)}{(\sec\theta + \tan\theta)^2} = \dfrac{\sec\theta}{\sec\theta + \tan\theta}$

PART - II

1. (a) $\int \csc x \cot x\, dx$

$- \csc x + C$

(b) $\int -\csc 5x \cot 5x\, dx$

$= \dfrac{\csc 5x}{5} + C$

(c) $\int -\pi\csc \dfrac{\pi}{2} x \cot \dfrac{\pi x}{2}\, dx$

$= \dfrac{+\pi\csc \dfrac{\pi}{2}x}{\pi/2} = 2\csc\dfrac{\pi}{2}x + C$

2. $\int (1 + 2\cos x)^2\, dx$

$= \int (1 + 4\cos^2 x + 4\cos x)dx$

$= \int 1.dx + \int 4\cos x\, dx + \int 4\cos^2 x\, dx$

$= x + 4\sin x + \int 4\left(\dfrac{\cos 2x + 1}{2}\right)dx$

$= x + 4\sin x + \int 2\cos 2x + \int 2 \cdot dx$

$= x + 4\sin x + \dfrac{2\sin 2x}{2} + 2x$

$= 3x + 4\sin x + \sin 2x + C$

3. $\int \dfrac{x^{1/2}}{2}dx + \int 2x^{-1/2}dx$

$= \dfrac{1}{2}\dfrac{x^{1/2+1}}{(1/2+1)} + 2\dfrac{x^{-1/2+1}}{-\dfrac{1}{2}+1} + c = \dfrac{1}{3}x^{3/2} + 4x^{1/2} + c$

4. $\int 8y\, dy - \int \dfrac{2}{y^{1/4}}dy$

$\dfrac{8y^2}{2} - \dfrac{2\ y^{-1/4+1}}{-\dfrac{1}{4}+1} + c$

$4y^2 - \dfrac{8}{3}y^{3/4} + c$

5. $\int [(2x - (2x)x^{-3})]dx$

$= \int 2x\, dx - \int 2x^{-2}dx$

$= \dfrac{2x^2}{2} - \dfrac{2x^{-2+1}}{-2+1} = x^2 + \dfrac{2}{x} + c$

10. $\int 4\sec x \tan x \; dx - 2\int \sec^2 x \; dx$

= 4 secx − 2 tanx + c

11. $\int \dfrac{1}{2}\csc^2 x \; dx - \dfrac{1}{2}\int \csc x \cot x \; dx$

$= -\dfrac{1}{2}\cot x + \dfrac{1}{2}\csc x + c$

12. $\int \sin 2x\, dx - \int \csc^2 x\, dx$

$= -\dfrac{\cos 2x}{2} + \cot x + c$

13. $\int 2\cos 2x\, dx - \int 3\sin 3x\, dx$

$\dfrac{2\sin 2x}{2} - 3\left(\dfrac{-\cos 3x}{3}\right) + c$

sin 2x + cos 3x + c

14. $\int 4\sin^2 y\, dy$

$\therefore \qquad \sin^2 y = \dfrac{1-\cos 2y}{2}$

$\int 4\left(\dfrac{1-\cos 2y}{2}\right)dy$

$= \int 2\,dy - \int 2\cos 2y\, dy$

$= 2y - \dfrac{2\sin 2y}{2}$

= 2y − sin 2y + c

15. $\int \dfrac{\csc \theta}{\csc \theta - \sin \theta}\, d\theta$

$= \int \dfrac{\dfrac{1}{\sin\theta}}{\dfrac{1}{\sin\theta} - \sin\theta}\, d\theta$

$= \int \dfrac{1}{1-\sin^2\theta}\, d\theta = \int \sec^2\theta \cdot d\theta = \tan\theta + c$

16. $\int \dfrac{1}{\sqrt{5s+4}}\, ds$

Let u = 5s + 4
du = 5 ds

$ds = \dfrac{du}{5}$

$= \dfrac{1}{5}\int u^{-1/2}du = \dfrac{1}{5}\dfrac{u^{-1/2+1}}{-1/2+1} + c$

$= \dfrac{2}{5}\sqrt{u} + c = \dfrac{2}{5}\sqrt{5s+4} + c$

17. $\int 3y \ \sqrt{7-3y^2} \ dy$

Let $7 - 3y^2 = t^2$

diff. w.r.t. y both side

$- 6y \ dy = 2t \ dt$

$= 3y \ dy = - t \ dt$

$= \int -t^2 \ dt$

$= - \dfrac{t^3}{3} + c$

$= - \dfrac{(7-3y^2)^{3/2}}{3} + c$

18. $\int \sin^5 \dfrac{x}{3} \cos \dfrac{x}{3} dx$

Let $\sin \dfrac{x}{3} = t$

$\cos \dfrac{x}{3} \dfrac{dx}{3} = dt$

$\cos \dfrac{x}{3} dx = 3 \ dt$

$= \int t^5 3dt$

$= 3\dfrac{t^6}{6} + c = \dfrac{\sin^6 \dfrac{x}{3}}{2} + c$

19. $\int \tan^7 \left(\dfrac{x}{2} \right) \sec^2 \dfrac{x}{2} \cdot dx$

Let $\tan \dfrac{x}{2} = t$

$\sec^2 \dfrac{x}{2} \cdot \dfrac{dx}{2} = dt$

$\sec^2 \dfrac{x}{2} \ dx = 2 \ dt$

$\int t^7 \ 2dt = 2 \dfrac{t^8}{8} + c = \dfrac{\tan^8 x/2}{4} + c$

20. $\int r^2 \left(\dfrac{r^3}{18} - 1 \right)^5 dr$

Let $\dfrac{r^3}{18} - 1 = t$

$\dfrac{3r^2}{18} dr = dt$

$r^2 dr = 6dt$

$= \int t^5 6 \ dt$

$= 6\dfrac{t^6}{6} + c = \left(\dfrac{r^3}{18} - 1 \right)^6 + c$

21. $\int r^4 \left(7 - \dfrac{r^5}{10}\right)^3 dr$

Let $7 - \dfrac{r^5}{10} = t$

$-\dfrac{5r^4}{10} \, dr = dt$

$r^4 \, dr = -2 \, dt$

$\int t^3 (-2dt)$

$= -2 \dfrac{t^4}{4} + c$

$= -\dfrac{\left(7 - \dfrac{r^5}{10}\right)^4}{2} + c$

22. $\int x^{\frac{1}{3}} \sin(x^{\frac{4}{3}} - 8) \, dx$

Let $x^{4/3} - 8 = t$

$\dfrac{4}{3} x^{1/3} \, dx = dt$

$x^{1/3} \, dx = \dfrac{4}{3} dt$

$\int \dfrac{3}{4} \sin t \, dt$

$= -\dfrac{3}{4} \cos t + c$

$= -\dfrac{3}{4} \cos (x^{4/3} - 8) + c$

23. $\int \csc\left(\dfrac{\upsilon - \pi}{2}\right) \cot\left(\dfrac{\upsilon - \pi}{2}\right) d\upsilon$

Let $\dfrac{\upsilon - \pi}{2} = t \Rightarrow d\upsilon = 2dt$

$\int \csc t \cot t \, 2dt = -2 \csc t$

$= -2 \csc\left(\dfrac{\upsilon - \pi}{2}\right) + c$

24. $\int \sqrt{\cot y} \ \ \csc^2 y \ dy$

Let $\cot y = t$

$-\csc^2 y \, dy = dt$

$-\int \sqrt{t} \, dt$

$-\dfrac{2t^{3/2}}{3} + c$

$= -\dfrac{2}{3} (\cot y)^{3/2} + c$

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25. $\int \dfrac{\sec z \tan z}{\sqrt{\sec z}} \, dr$

Let $\sec z = t$

$\sec z \tan z \, dz = dt$

$\int \dfrac{1}{\sqrt{t}} \, dt$

$= 2\sqrt{t} + c$

$= 2\sqrt{\sec z} + c$

26. $\int \dfrac{1}{t^2} \cos\left(\dfrac{1}{t} - 1\right) dt$

Let $\dfrac{1}{t} - 1 = x$

$= -\dfrac{1}{t^2} \, dt = dx$

$\int \cos x(-dx)$

$= -\sin x + c$

$= -\sin\left(\dfrac{1}{t} - 1\right) + c$

27. $\int\limits_{1/2}^{3/2} (-2x + 4) \, dx$

$= \left[-x^2 + 4x\right]_{1/2}^{3/2}$

$= \left[-\left(\dfrac{3}{2}\right)^2 + 4\left(\dfrac{3}{2}\right)\right] - \left[-\dfrac{1}{4} + 4 \times \dfrac{1}{2}\right]$

$= 2$ square units

28. $\int\limits_{-2}^{1} |x| \, dx$

$\int\limits_{-2}^{0} -x \, dx + \int\limits_{0}^{1} dx = \left[-\dfrac{x^2}{2}\right]_{-2}^{0} + \left[\left(\dfrac{x^2}{2}\right)\right]_{0}^{1}$

$= 2 + 1/2 = 2.5$ square units

29. $\int\limits_{0}^{\pi/2} \theta^2 \, d\theta \qquad \Rightarrow \qquad \left[\dfrac{\theta^3}{3}\right]_{0}^{\pi/2} \qquad \Rightarrow \qquad \dfrac{\pi^3}{24}$

30. $\int\limits_{0}^{3b} x^2 \, dx \qquad \Rightarrow \qquad \left[\dfrac{x^3}{3}\right]_{0}^{3b} = 9b^3$

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PART - III

1.
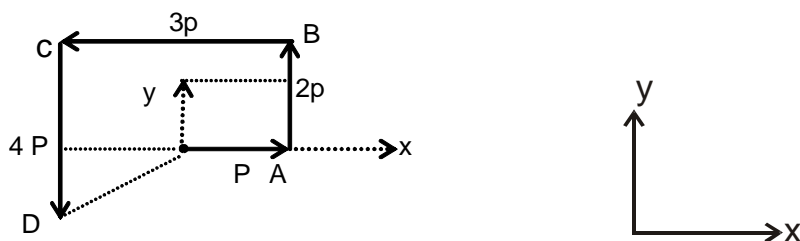
Let O be the origin & unit vector along x axis is $\hat{i}$ & unit vector along y axis is $\hat{j}$

So, $\overrightarrow{OA} = P\hat{i}$

$\overrightarrow{AB} = 2P\hat{j}$

$\overrightarrow{BC} = -3P\hat{i}$

$\overrightarrow{CD} = -4P\hat{j}$

Resultant force

$\vec{R} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$

$= P\hat{i} + 2p\hat{i} - 3p\hat{i} - 4p\hat{j}$

$\vec{R} = -2p\hat{i} - 2p\hat{j}$

$|\vec{R}| = 2\sqrt{2}$ P **Ans.**

2.
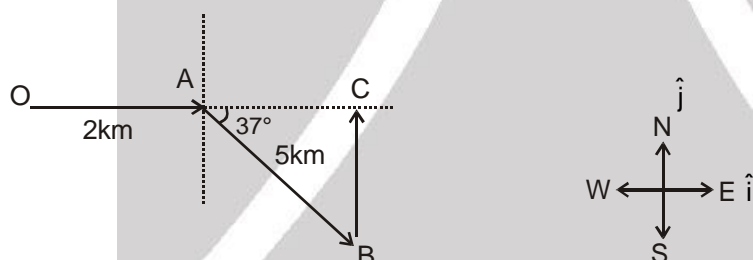
Final displacement of boat = $6\hat{i}$, from figure

$\overrightarrow{OA} = 2\hat{i}$

$\overrightarrow{BC} = 5\cos37°\,\hat{i} - 5\sin37°\,\hat{j} = 4\hat{i} - 3\hat{j}$

from figure

$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC}$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OA} - \overrightarrow{AB}$

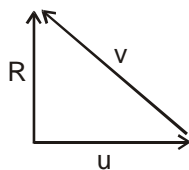$= 6\hat{i} - 2\hat{i} - 4\hat{i} + 3\hat{j}$

$\overrightarrow{BC} = 3\hat{j}$

= 3 km in north

3. Given $\vec{R} = \vec{u} + \vec{v}$ and $\vec{R} \perp \vec{u}$

and $|\vec{R}| = \dfrac{|\vec{v}|}{2}$

Let $|\vec{v}| = x$ then $|\vec{R}| = x/2$

from figure $u^2 = v^2 - R^2 = x^2 - \dfrac{x^2}{4} \Rightarrow u = \dfrac{\sqrt{3}}{2} x$
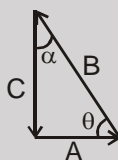
we know that

$R^2 = u^2 + v^2 + 2uv\cos\theta$

$\dfrac{x^2}{4} = \dfrac{3}{4} x^2 + x^2 + 2\left(\dfrac{\sqrt{3}}{2} x \right) (x)\cos\theta$

$\Rightarrow \qquad \cos\theta = -\dfrac{\sqrt{3}}{2}$

$\qquad\qquad \theta = 150°$

4. Given that $|\vec{A}| = 5N$, $|\vec{B}| = 13N$ & $|\vec{C}| = 12N$

for angle between $\vec{A}$ & $\vec{B}$, from figure $\sin\alpha = \dfrac{5}{13}$

$\alpha = 23°$
so angle $\theta = 90 - 23$
so angle between $\vec{A}$ & $\vec{B} = 180 - \theta = 180 - 90 + 23 = 90 + 23 = 113°$

5. From given condition
$|\vec{A} + \vec{B}| = |\vec{A}| = |\vec{B}| = x$

$A^2 + B^2 + 2AB\cos\theta = A^2$

$x^2 + x^2 + 2x^2\cos\theta = x^2$
$\cos\theta = -1/2$
$\theta = 120°$
$|\vec{A} - \vec{B}| = |\vec{A}|$

$A^2 + B^2 - 2AB\cos\theta = A^2$

$x^2 + x^2 - 2x^2\cos\theta' = x^2$

$\cos\theta' = 1/2$

$\theta' = 60°$

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6. Let θ be the angle between $\vec{P}$ and $\vec{Q}$. Then.

$$R^2 = |\vec{P} + \vec{Q}|^2 = P^2 + Q^2 + 2PQ \cos \theta \qquad ....(i)$$

If $\vec{Q}$ is doubled, $\vec{R}$ is doubled. That means, the magnitude of resultant of $2\vec{Q}$ and $\vec{P}$ is 2R

$$(2R)^2 = P^2 + (2Q)^2 + 2P (2Q) \cos\theta$$

This yields, $\qquad 4R^2 = P^2 + 4Q + 4PQ \cos\theta \qquad ....(ii)$

When $\vec{Q}$ is reversed, $\vec{R}$ is doubled. Hence, the magnitude of resultant of $\vec{P}$ and $(-\vec{Q})$ is 2R.

Then, $\qquad\qquad\qquad (2R)^2 = P^2 + Q^2 + 2PQ \cos (180^0 - \theta).$

This yields $\qquad\qquad 4R^2 = P^2 + Q^2 - 2PQ\cos\theta \qquad ...(iii)$

Eq. (i) – Eq. (iii) yields $\qquad PQ \cos \theta = \dfrac{-3R^2}{4} \qquad ...(iv)$

Eq. (i) + Eq. (iii) yields $\qquad P^2 + Q^2 = \dfrac{5R^2}{2} \qquad ....(v)$

Eq. (ii) + Eq (iv) yields $\qquad P^2 + 4Q^2 = 7R^2 \qquad ...(vi)$

Solving Eq. (v) and Eq. (vi) we obtain $Q = \sqrt{\dfrac{3}{2}}$ R and P = R

Hence $P : Q : R = \sqrt{2} : \sqrt{3} : \sqrt{2}$ **Ans.**

7. It is relevant from the vector diagram that each vector deviates from its neighbour by an angle of 60⁰. We bring the tails of each vector to a point (origin) and observe that $\vec{A}_1$ and $\vec{A}_4$ are equal and opposite; and $\vec{A}_2$ and $\vec{A}_5$ are equal and opposite.
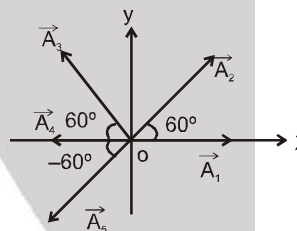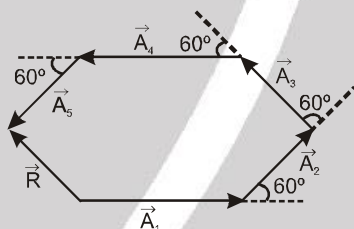
$\vec{A}_1 + \vec{A}_4 = \vec{0}$ and $\vec{A}_2 + \vec{A}_5 = \vec{0}$

Hence $\vec{R} = \vec{A}_1 + \vec{A}_2 + \vec{A}_3 + \vec{A}_4 + \vec{A}_5$

$= \vec{A}_3 = (|\vec{A}_3| \cos 60^0) (-\hat{i}) + (|\vec{A}_3| \sin 60^0) \hat{j}$

Substituting $|\vec{A}_3| = 1$, we have, $\vec{A}_3 = \dfrac{1}{2}(-\hat{i} + \sqrt{3}\hat{j})$

8. The forces exerted by the ants are given as,

$$\vec{F}_1 = 3 (\cos 30^0 \, \hat{i} + \sin 30^0 \hat{j})$$

$$\vec{F}_2 = -1\hat{i}$$

$$\vec{F}_3 = 2 \cos 45^0 \hat{i} + 2 \sin 45^0 (-\hat{j})$$
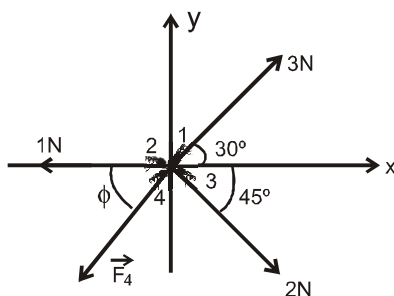
and (= F) = $x\hat{i} + y\hat{j}$, say.

Since the grain is in equilibrium, it experiences a net force

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_4 = \vec{0}$$

Substituting the values of the above forces, we have

$$\hat{i}\left(\frac{3\sqrt{3}}{2} - 1 + \sqrt{2} + x\right) + \left(\frac{3}{2} - \sqrt{2} + y\right)\hat{j} = 0$$

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Comparing the coefficients of $\hat{i}$ and $\hat{j}$, we have

$x = -\left(\dfrac{3\sqrt{3}}{2} + \sqrt{2} - 1\right)$ and $y = -\left(\dfrac{3}{2} - \sqrt{2}\right)$
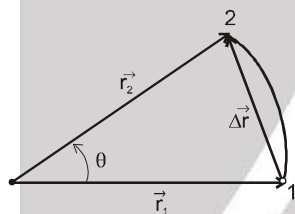
Hence the force exerted by the 4th ant has the magnitude

$$F_4 = \sqrt{x^2 + y^2} = \sqrt{\left(\dfrac{3\sqrt{3}}{2} + \sqrt{2} - 1\right)^2 + \left(\dfrac{3}{2} - \sqrt{2}\right)^2}$$

and directed at an angle $\phi = \tan^{-1}\left[\dfrac{3 - 2\sqrt{2}}{3\sqrt{3} + 2\sqrt{2} - 2}\right]$ with –ve x-direction, as shown in the figure.

9. When the insect moves from position 1 to position 2, the displacement $\vec{s}$ = change in position vector $\Delta\vec{r}$. Since $\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$, the magnitude of the displacement is
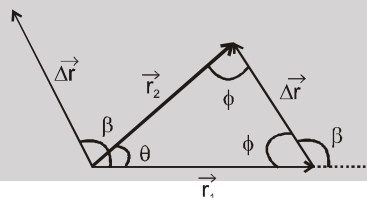
$|\Delta\vec{r}| = \sqrt{r_1^2 + r_2^2 - 2r_1\ r_2 \cos\theta}$

Since the insect moves in a circular path,
$r_1 = r_2 = R$

Hence $|\Delta\vec{r}| = 2R\sin\theta/2$. The direction of $\Delta\vec{r}$ is given as $\beta = \dfrac{\pi}{2} + \dfrac{\theta}{2}$, as shown in the figure.

10. Let us draw a straight line PQ which is given by

$$\vec{PQ} = \vec{OQ} - \vec{OP}$$

where $\vec{OQ} = \vec{r}$ and $\vec{OP} = \vec{r}_0$ (given)
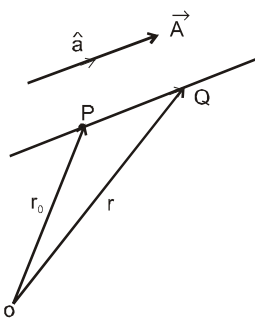
This gives $\vec{PQ} = \vec{r} - \vec{r}_0$

Since $\vec{PQ} \parallel \vec{A}$ (given)

$\vec{PQ} = PQ.\hat{a}$, where $\hat{a} = \dfrac{\vec{A}}{A}$

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Hence $\vec{r} - \vec{r}_0 = \dfrac{PQ}{A}\hat{a}$ This yields

$\vec{r} = \vec{r}_0 + \dfrac{PQ}{A}\hat{a}$. Then by putting $\dfrac{PQ}{A} = n$, we obtain $\vec{r} = \vec{r}_0 + n\hat{a}$

11. $\vec{A} + \vec{B} = \vec{R}$, say,

Since $|\vec{R}|^2 = \vec{R}.\vec{R} = (\vec{A} + \vec{B}).(\vec{A} + \vec{B})$

expanding the product, we have

$|\vec{R}|^2 = \vec{A}.\vec{A} + \vec{A}.\vec{B} + \vec{B}.\vec{A} + \vec{B}.\vec{B}$

Substituting $\vec{A}.\vec{A} = A^2$, $\vec{A}.\vec{B} = \vec{B}.\vec{A}$ and $\vec{B}.\vec{B} = |\vec{B}|^2$, we have

$|\vec{R}|^2 = |\vec{A}|^2 + |\vec{B}|^2 + 2\vec{A}.\vec{B}$

Hence $|\vec{R}| = |\vec{A} + \vec{B}| = \sqrt{|\vec{A}|^2 + |\vec{B}|^2 + 2|\vec{A}||\vec{B}|\cos\theta}$

12. Differentiating both sides of $\vec{A} = t\hat{i} - \sin\pi t\hat{j} + t^2\hat{k}$

we have $\dfrac{d\vec{A}}{dt} = \dfrac{d}{dt}(t\hat{i} - \sin\pi t\hat{j} + t^2\hat{k})$

$= \dfrac{d}{dt}(t)\hat{i} - \dfrac{d}{dt}(\sin\pi t)\hat{j} + \dfrac{d}{dt}(t^2)\hat{k}$

$= \hat{i} - \pi\cos\pi t\hat{j} + 2t\hat{k}$

Putting t = 1, we have $\left.\dfrac{d\vec{A}}{dt}\right|_{t=1} = \hat{i} - (\pi\cos\pi)\hat{j} + 2(1)\hat{k}$

$= \hat{i} + \pi\hat{j} + 2\hat{k}$

13. We have

Position : s = 5 cos t

Velocity : v = $\dfrac{ds}{dt} = \dfrac{d}{dt}$ (5 cos t) = $5\dfrac{d}{dt}$ (cos t) = $-$ 5 sin t

Acceleration : a = $\dfrac{dv}{dt} = \dfrac{d}{dt}$ ($-$ 5 sin t) = $-5\dfrac{d}{dt}$ (sin t) = $-$ 5 cos t

14. (a) volume v of water remains constant

$v = \pi (4R^2) h - \pi R^2 x$

$0 = 16\pi R^2 \dfrac{dh}{dt} - \pi R^2 \dfrac{dx}{dt} = 0$

$\dfrac{dh}{dt} = \dfrac{1}{16} v$

(b) Area of wet surface ℓ

$A = \pi R^2 + 2\pi R \dfrac{d\ell}{dt}$

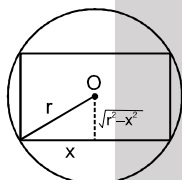ℓ is length inside water

$\dfrac{dA}{dt} = 0 + 2\pi R \dfrac{d\ell}{dt}$ ∵ $\dfrac{d\ell}{dt} = \dfrac{dx}{dt} + \dfrac{dh}{dt}$

$\dfrac{dA}{dt} = 2\pi R \left[v + \dfrac{v}{16} \right] = \dfrac{17\pi}{8}\ Rv$

Ans. (a) $\dfrac{dh}{dt} = \dfrac{v}{15}$ (b) $\dfrac{32\pi Rv}{15}$

15.

Let A = area of rectangle

$A = 8 \times \dfrac{1}{2} \times x \times \sqrt{r^2 - x^2} = 4\,x\,\sqrt{r^2 - x^2}$

For maxima :

$\dfrac{dA}{dx} = 0, \quad \dfrac{dA}{dx} = 4\left(\dfrac{x(-2x)}{2\sqrt{r^2 - x^2}} + \sqrt{r^2 - x^2} \right)$

$0 = \dfrac{4(r^2 - 2x^2)}{\sqrt{r^2 - x^2}}$

or $x = \dfrac{r}{\sqrt{2}}$

For $x = \dfrac{r}{\sqrt{2}}, \qquad \dfrac{d^2 A}{dx^2} < 0$

∴ Area is maximum for $x = \dfrac{r}{\sqrt{2}}$

and $A_{max} = \dfrac{4r}{\sqrt{2}} \sqrt{r^2 - \left(\dfrac{r}{\sqrt{2}} \right)^2}$

$A_{max} = 2r^2.$